# Supplementary Note for New Double Soft Emission Theorems

## Proofs Up to the Sub-Leading Order

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We present detailed proofs of two classes of double soft theorems for scalar theories, using the integral representation based on scattering equations. We also extend the discussion to more general cases at the end, including an additional class of theorems for simultaneous emission of two soft photons.

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The use of scattering equations leads to an integral representation for amplitudes in a large variety of theories of massless bosons. A particularly nice feature is that it allows a straightforward analysis of single soft particle emission in theories like Yang–Mills and Einstein gravity [1]. In a recent work [2] we found two classes of soft theorems for simultaneous emission of two soft scalars. In this supplementary note, we provide detailed proofs for these new theorems following the same spirit.

Let us first quickly review this integral representation, which has the following general form

$$M_N = \int \frac{\prod_{a=1}^{N} d\sigma_a\, \delta(f_a)}{\text{(redundancies)}}\, I_N(k,\epsilon,\sigma) =: \int d\mu_N\, I_N\,, \tag{1}$$

where $\sigma$'s are holomorphic variables specifying the locations of punctures on an auxiliary Riemann sphere, and

$$f_a := \sum_{\substack{b=1\\ b\neq a}}^{N} \frac{k_a\cdot k_b}{\sigma_a-\sigma_b}\,, \quad \forall a\,, \tag{2}$$

impose the so-called scattering equations, and the integrand $I_n$ is some rational function of the kinematics data and the $\sigma$ variables. Due to the redundancies in both the variables and the equations, the correct way to implement (1) is to choose any two sets of labels $\{a,b,c\}$ and $\{a',b',c'\}$, delete $d\sigma_a\, d\sigma_b\, d\sigma_c\, \delta(f_{a'})\, \delta(f_{b'})\, \delta(f_{c'})$, compensated by a factor $(\sigma_{ab}\, \sigma_{bc}\, \sigma_{ca}\, \sigma_{a'b'}\, \sigma_{b'c'}\, \sigma_{c'a'})$ (we abbreviate, e.g., $\sigma_{ab} := \sigma_a - \sigma_b$), and the result is independent of the choice.

While the integration is performed over the moduli space of the $N$-punctured Riemann spheres, it is fully localized by the scattering equations onto their solutions. Hence genuinely $M_N$ is a summation

$$M_N = \sum_{\sigma \text{ solutions}} \frac{I_N}{J_N}\,, \tag{3}$$

where $J_N$ is the Jacobian from solving the constraints in (1).

The only part in (1) that depends on specific theories is $I_N$, which can be constructed from simple building blocks. If we only study scalar amplitudes, there are three building blocks of interests: (i) the Parke–Taylor factor

$$C(\alpha) := \frac{1}{\sigma_{\alpha(1)\alpha(2)}\, \sigma_{\alpha(2)\alpha(3)} \cdots \sigma_{\alpha(N)\alpha(1)}}\,, \tag{4}$$

where $\alpha$ denotes a given ordering, (ii) the Pfaffian $\mathrm{Pf}X_N$, and (iii) the reduced Pfaffian $\mathrm{Pf}'A_N$, where the two $N\times N$ anti-symmetric matrices are defined respectively by their entries as

$$(X_N)_{ab} := \frac{1}{\sigma_a-\sigma_b}\,(1-\delta_{ab})\,, \quad (A_N)_{ab} := \frac{k_a\cdot k_b}{\sigma_a-\sigma_b}\,(1-\delta_{ab})\,. \tag{5}$$

Since matrix $A_N$ has corank 2 on the support of the scattering equations $f_a = 0$ $(\forall a)$, we define the invariant quantity $\mathrm{Pf}'A_N := \frac{(-1)^{a+b}}{\sigma_a-\sigma_b}\mathrm{Pf}[A_N]_{\hat{a},\hat{b}}$, where the minor $[A_N]_{\hat{a},\hat{b}}$ is obtained by deleting rows and columns labeled by any given $\{a,b\}$.

The theories we are interested in are a special Galileon with enhanced symmetry (sGal), Dirac–Born–Infeld (DBI), Einstein–Maxwell–Scalar (EMS), the $U(N)$ non-linear sigma model (NLSM), and Yang–Mills–Scalar (YMS) [3]. With these building blocks, the integrands for scalars amplitudes in these theories are constructed as follows

$$\begin{aligned} &I_N^{\text{sGal}} := (\text{Pf}' A_N)^4\,, \quad I_N^{\text{DBI}} := \text{Pf}X_N\,(\text{Pf}' A_N)^3\,, \quad I_N^{\text{EMS}} := (\text{Pf}X_N)^2\,(\text{Pf}' A_N)^2\,, \\ &I_N^{\text{NLSM}}(\alpha) := C(\alpha)\,(\text{Pf}' A_N)^2\,, \quad I_N^{\text{YMS}}(\alpha) := C(\alpha)\,\text{Pf}X_N\,\text{Pf}' A_N\,. \end{aligned} \tag{6}$$

Here we choose to normalize the amplitude such that there is no extra constant overall factor in the formula above. Hence this might differ by some overall constant from the expression directly obtained from Feynman diagram calculation using the standard Lagrangian for these theories. But they can always be related by a re-definition of the coupling.

## 2 General prescription

In this section we describe the general prescription for analyzing the double soft limit with any formula in the representation based on scattering equations.

Let us first fix the convention. We consider the double soft limit of an $N$-point amplitude, and denote the number of particles in the lower-point amplitude as $n$, i.e., $N = n + 2$. Without loss of generality, we assume particles labeled by $n + 1$ and $n + 2$ to be soft. This is achieved by introducing a soft parameter $\tau$

$$k_{n+1} = \tau\, p\,, \quad k_{n+2} = \tau\, q\,, \tag{7}$$

and probing the limit $\tau \to 0$ (with $p$ and $q$ fixed).

Starting with the formula for the higher-point amplitude $M_N$, let us not delete $d\sigma_{n+1}\, d\sigma_{n+2}$ nor $\delta(f_{n+1})\,\delta(f_{n+2})$. It is nice to work with a new set of variables

$$\sigma_{n+1} = \rho - \frac{\xi}{2}\,, \quad \sigma_{n+2} = \rho + \frac{\xi}{2}\,, \tag{8}$$

and do the transformation (we do not care a possible overall sign)

$$d\sigma_{n+1}\, d\sigma_{n+2}\, \delta(f_{n+1})\, \delta(f_{n+2}) = -\, d\rho\, d\xi\, 2\, \delta(f_{n+1} + f_{n+2})\, \delta(f_{n+1} - f_{n+2})\,. \tag{9}$$

Clearly, we need to get rid of the $\rho$ and $\xi$ integrations in order to land on the lower-point amplitude. The general idea is to localize the $\xi$ integration using $\delta(f_{n+1} - f_{n+2})$, and regard the $\rho$ integration as a contour integration whose contour wraps the zeros of $f_{n+1} + f_{n+2}$. This leads to

$$M_N = \oint \frac{d\rho}{2\pi i} \sum_{\xi \text{ solutions}} \int d\mu'_n\, \frac{1}{f_{n+1} + f_{n+2}}\, \frac{-2}{\frac{\partial}{\partial \xi}\,[f_{n+1} - f_{n+2}]}\, I_N(k, \sigma, \rho, \xi)\,, \tag{10}$$

where $\xi$ is understood to be evaluated on its solutions to the equation $f_{n+1} - f_{n+2} = 0$. We also use the notation $d\mu'_n$ instead of the measure $d\mu_n$ for a genuine $n$-point amplitude to indicate that $f_a$'s in the remaining delta constraints are still the ones for an $N$-point amplitude. Instead, for those scattering equations entering the formula for the lower-point amplitude (i.e., those in $d\mu_n$) we choose to denote them by $g_a$ in order to avoid confusion.

As we are going to see, in general the remaining $\rho$ integration can be easily computed by an application of the global residue theorem, the result of which can be directly interpreted as an operation acting on the formula for the corresponding lower-point amplitude.

We first discuss the behavior of different parts of the formula.

With the new variables, the functions entering the delta constraints explicitly read

$$f_a = \begin{cases} \sum_{\substack{b=1\\ b\neq a}}^n \frac{k_a\cdot k_b}{\sigma_a-\sigma_b} + \frac{\tau\, k_a\cdot p}{\sigma_a-\rho+\frac{\xi}{2}} + \frac{\tau\, k_a\cdot q}{\sigma_a-\rho-\frac{\xi}{2}}\,, & a\neq n+1, n+2\,,\\ \sum_{b=1}^n \frac{\tau\, k_b\cdot p}{\rho-\frac{\xi}{2}-\sigma_b} + \frac{\tau^2\, p\cdot q}{-\xi}\,, & a=n+1\,,\\ \sum_{b=1}^n \frac{\tau\, k_b\cdot q}{\rho+\frac{\xi}{2}-\sigma_b} + \frac{\tau^2\, p\cdot q}{\xi}\,, & a=n+2\,. \end{cases} \tag{11}$$

Thus (10) has the form

$$M_N = \oint \frac{d\rho}{2\pi i} \sum_{\xi \text{ solns}} \int d\mu_n' \frac{1}{\sum_{b=1}^n \left(\frac{k_b\cdot p}{\rho-\frac{\xi}{2}-\sigma_b} + \frac{k_b\cdot q}{\rho+\frac{\xi}{2}-\sigma_b}\right)} \frac{-4\,\tau^{-2}}{\sum_{b=1}^n \left(\frac{k_b\cdot p}{(\rho-\frac{\xi}{2}-\sigma_b)^2} + \frac{k_b\cdot q}{(\rho+\frac{\xi}{2}-\sigma_b)^2}\right) + \frac{4\,\tau\, p\cdot q}{\xi^2}} I_N\,. \tag{12}$$

Let us have a close look at the equation that determine the solutions of $\xi$, i.e.,

$$f_{n+1} - f_{n+2} = \sum_{b=1}^n \left(\frac{\tau\, k_b\cdot p}{\rho-\frac{\xi}{2}-\sigma_b} - \frac{\tau\, k_b\cdot q}{\rho+\frac{\xi}{2}-\sigma_b}\right) - \frac{2\,\tau^2\, p\cdot q}{\xi} = 0\,. \tag{13}$$

Since $\rho$ is the average of $\sigma_{n+1}$ and $\sigma_{n+2}$, as long as a proper gauge is fixed for the $SL(2,\mathbb{C})$ redundancy among the $\sigma$ variables we expect $\rho$ to be finite on the support of the scattering equations. In this region of $\rho$, when $\tau\to 0$ the $\xi$ solution to (13) fall into two types: the non-degenerate solutions ($\xi\sim\tau^0$) and a unique degenerate solution ($\xi\sim\tau^1$). As was summarized in [2], for the scalar amplitudes under study both the leading and sub-leading contributions are received from the degenerate solution only. Hence in (12) we are justified to ignore all the non-degenerate $\xi$ solutions. In this case we can perturbatively expand $\xi$ as

$$\xi = \tau\,\xi_1 + \tau^2\,\xi_2 + \mathcal{O}(\tau^3), \tag{14}$$

whose explicit solution is

$$\frac{1}{\xi_1} = \frac{1}{2\,p\cdot q}\sum_{b=1}^n \frac{k_b\cdot(p-q)}{\rho-\sigma_b}\,, \tag{15}$$

$$\frac{\xi_2}{\xi_1^3} = -\frac{1}{4\,p\cdot q}\sum_{b=1}^n \frac{k_b\cdot(p+q)}{(\rho-\sigma_b)^2}\,. \tag{16}$$

Then (12) can be further expanded to

$$M_N = \oint \frac{d\rho}{2\pi i} \int d\mu_n' \frac{1}{\sum_{b=1}^n \frac{k_b\cdot(p+q)}{\rho-\sigma_b}} \frac{-\xi_1^2\, I_N}{\tau\, p\cdot q} \left(1 - \frac{\tau\,\xi_1}{2} \frac{\sum_{b=1}^n \frac{k_b\cdot(p-q)}{(\rho-\sigma_b)^2}}{\sum_{b=1}^n \frac{k_b\cdot(p+q)}{\rho-\sigma_b}} + 3\,\tau\,\frac{\xi_2}{\xi_1} + \mathcal{O}(\tau^2)\right), \tag{17}$$

where now the $\rho$ contour wraps around the zeros of $\sum_{b=1}^n \frac{k_b\cdot(p+q)}{\rho-\sigma_b}$.

To extract a given order in $\tau$ we still need to expand $d\mu'_n$ and $I_N$. We are going to discuss $I_N$ in the next subsection. As mentioned before $d\mu'_n$ differs from $d\mu_n$ in that each $f_a$ ($a = 1, \ldots, n$) in the delta constraints are for the higher-point amplitude. So we can expand the measure as

$$d\mu'_n = d\mu_n + {\sum_{a=1}^{n}}' \frac{d\mu_n}{\delta(g_a)}\, \delta'(g_a)\, \frac{\tau\, k_a\cdot(p+q)}{\sigma_a - \rho} + \mathcal{O}(\tau^2)\,, \quad g_a := \sum_{\substack{b=1\\ b\neq a}}^{n} \frac{k_a\cdot k_b}{\sigma_a-\sigma_b}\,, \tag{18}$$

where (as mentioned before) $\delta(g_a)$'s are the constraints entering the lower-point amplitude, and $\sum'$ means the summation does not include the three labels for constraints that we deleted in fixing the redundancies.

At first sight the appearance of $\delta'$ might be worrisome, but later on we are going to show that the contribution of such term to the sub-leading order can be exactly mapped to the result of $S^{(1)}$ acting on $\delta(g_a)$ in the lower-point formula.

For the time being, let us have a look at the leading order in the double soft limit. Suppose the higher-point integrand $I_N$ behaves like

$$I_N(k,\sigma,\rho,\xi) = F(k,\sigma,\rho,\xi)\, I_n(k,\sigma) + (\text{sub-leading})\,, \tag{19}$$

where $I_n(k,\sigma)$ is exactly the integrand for the lower-point amplitude and $F$ some rational function, then by (17) and (18) we have

$$M_N = \int d\mu_n\, I_n \oint \frac{d\rho}{2\pi i}\, \frac{1}{\sum_{b=1}^{n} \frac{k_b\cdot(p+q)}{\rho-\sigma_b}}\, \frac{-\xi_1^2}{\tau\, p\cdot q}\, F(k,\sigma,\rho,\tau\,\xi_1) + (\text{sub-leading})\,. \tag{20}$$

We would thus expect

$$S^{*(0)} = \oint \frac{d\rho}{2\pi i}\, \frac{1}{\sum_{b=1}^{n} \frac{k_b\cdot(p+q)}{\rho-\sigma_b}}\, \frac{-\xi_1^2}{\tau\, p\cdot q}\, F(k,\sigma,\rho,\tau\,\xi_1) \tag{21}$$

to be the leading order soft factor, as long as it no longer has any $\sigma$ dependence. Here we put a star on $S^{(0)}$ to keep in mind that this multiplicative operator is obtained by counting the leading order effects only. As we are going to see, when taking sub-leading order into consideration this operator has to receive corrections.

However, there is a loophole in the above discussion because as we deform the $\rho$ contour in actual computation we are going to pass the neighborhood of $\rho = \infty$. It is fine as long as the formula does not have any pole at $\rho = \infty$, but if there is then extra attention has to be paid. Since we are keeping $\tau$ to be small but still finite, it is justified to assume $\rho\tau \gg 1$. In this region, we can check that any solution to (13) has to be $\xi \sim \rho$ to the leading order. Thus we assume an expansion of $\xi$ in terms of $\rho^{-1}$ instead of $\tau$, and at the leading order we find exactly two solutions

$$\xi = \pm 2\, i\, \rho + \mathcal{O}(\rho^0)\,. \tag{22}$$

Hence if we deform the contour in (12) to the one wrapping around $\rho = \infty$, to the leading order the resulting expression has the form

$$(M_N)_\infty = \oint \frac{d\rho}{2\pi i} \int d\mu_n \left( \frac{-4\,\rho^3}{\tau^4\,(p\cdot q)^2} + \mathcal{O}(\rho^2) \right) \left( I_N|_{\xi=2\,i\,\rho} + I_N|_{\xi=-2\,i\,\rho} + (\text{sub-leading}) \right)\,. \tag{23}$$

Before analyzing specific theories, here we also summarize the perturbative expansion of different building blocks that may enter into the integrand, following the prescription discussed so far. Since we are solely interested in scalar theories, there are altogether three building blocks that we need to care: the Parke–Taylor factor $C_N$, $\mathrm{Pf}X_N$, and $\mathrm{Pf}'A_N$. For simplicity we assume canonical ordering for $C_N$.

i). The expansion of the Parke–Taylor factor can be easily worked out

$$C_N = C_n \,\frac{(\sigma_n - \sigma_1)}{\tau\,(\sigma_n - \rho)\,(-\xi_1)\,(\rho - \sigma_1)} \left(1 - \frac{\tau\,\xi_1}{2\,(\sigma_n - \rho)} - \frac{\tau\,\xi_1}{2\,(\rho - \sigma_1)} - \frac{\tau\,\xi_2}{\xi_1} + \mathcal{O}(\tau^2)\right). \tag{24}$$

ii). For $\mathrm{Pf}X_N$, note that the structure of matrix $X_N$ is

$$X_N = \left(\begin{array}{c|cc} \frac{1}{\sigma_a-\sigma_b} & \frac{1}{\sigma_a-\rho+\frac{\xi}{2}} & \frac{1}{\sigma_a-\rho-\frac{\xi}{2}} \\ \hline \frac{1}{\rho-\frac{\xi}{2}-\sigma_b} & 0 & \frac{1}{-\xi} \\ \frac{1}{\rho+\frac{\xi}{2}-\sigma_b} & \frac{1}{\xi} & 0 \end{array}\right) \approx \left(\begin{array}{c|cc} \frac{1}{\sigma_a-\sigma_b} & \frac{1}{\sigma_a-\rho} & \frac{1}{\sigma_a-\rho} \\ \hline \frac{1}{\rho-\sigma_b} & 0 & \frac{1}{-\tau\,\xi_1-\tau^2\,\xi_2} \\ \frac{1}{\rho-\sigma_b} & \frac{1}{\tau\,\xi_1+\tau^2\,\xi_2} & 0 \end{array}\right), \tag{25}$$

where we made an approximation which is sufficient to account for all the sub-leading terms in $\mathrm{Pf}X_N$. Recall the definition of the Pfaffian of a $2\ell \times 2\ell$ anti-symmetric matrix $\mathbf{M}$

$$\mathrm{Pf}\mathbf{M} = \frac{1}{\ell!\,2^\ell} \sum_{\alpha \in S_{2\ell}} \mathrm{sgn}(\alpha)\, \mathbf{M}_{\alpha(1),\alpha(2)}\, \mathbf{M}_{\alpha(3),\alpha(4)} \cdots \mathbf{M}_{\alpha(2\ell-1),\alpha(2\ell)}\,, \quad \mathrm{sgn}(\alpha) := \begin{cases} +1\,, & \alpha \text{ even} \\ -1\,, & \alpha \text{ odd} \end{cases}. \tag{26}$$

Hence $\mathrm{Pf}X_N$ receives its leading contribution from terms involving $(X_N)_{n+1,n+2}$, and it has the following expansion up to the sub-leading order

$$\begin{aligned} \mathrm{Pf}X_N =& \frac{1}{-\tau\,\xi_1}\left(1 - \frac{\tau\,\xi_2}{\xi_1}\right)\mathrm{Pf}X_n \\ &+ \sum_{1\leq a<b\leq n} \frac{(-1)^{(n+2-b)+(n+1-a)-1}}{(\rho-\sigma_a)\,(\rho-\sigma_b)}\,\mathrm{Pf}[X_n]_{\hat{a},\hat{b}} + \sum_{1\leq b<a\leq n} \frac{(-1)^{(n+2-b)+(n+1-a)}}{(\rho-\sigma_a)\,(\rho-\sigma_b)}\,\mathrm{Pf}[X_n]_{\hat{b},\hat{a}} + \mathcal{O}(\tau)\,, \end{aligned} \tag{27}$$

where $[X_n]_{\hat{a},\hat{b}}$ is the minor of $X_n$ obtained by deleting rows and columns labeled by $a, b$. Obviously the two summations in (27) cancel each other and we are left with

$$\mathrm{Pf}X_N = \frac{1}{-\tau\,\xi_1}\left(1 - \frac{\tau\,\xi_2}{\xi_1}\right)\mathrm{Pf}X_n + \mathcal{O}(\tau)\,. \tag{28}$$

iii). The analysis on $\mathrm{Pf}'A_N$ is very similar to that on $\mathrm{Pf}X_N$, due to the similarity between the two matrices. However, the summation part in this case is non-trivial, and in the end we have

$$\begin{aligned} \mathrm{Pf}'A_N =& \frac{-\tau\,p\cdot q}{\xi_1}\left(1 - \frac{\tau\,\xi_2}{\xi_1}\right)\mathrm{Pf}'A_n \\ &+ \sum_{3\leq a<b\leq n}^{n} \frac{\tau^2\,(k_a\cdot p\;k_b\cdot q - k_a\cdot q\;k_b\cdot p)}{(\rho-\sigma_a)\,(\rho-\sigma_b)}\,\frac{(-1)^{a+b+1}\,\mathrm{Pf}[A_n]_{\hat{1},\hat{2},\hat{a},\hat{b}}}{(\sigma_1-\sigma_2)} + \mathcal{O}(\tau^3)\,. \end{aligned} \tag{29}$$

Finally, let us also have a quick look at the behavior of these building blocks in the neighborhood of $\rho = \infty$. Recall (22), the analysis is very similar to the above, and we directly list out the result up to the leading order (which is sufficient for the discussion in this note)

$$C_N = C_n \, \frac{(\sigma_n - \sigma_1)}{\pm 4\, i\, \rho^3} + (\text{sub-leading})\,, \tag{30}$$

$$\text{Pf}X_N = \frac{1}{\pm 2\, i\, \rho} \, \text{Pf}X_n + (\text{sub-leading})\,, \tag{31}$$

$$\text{Pf}' A_N = \frac{\tau^2 \, p \cdot q}{\pm 2\, i\, \rho} \, \text{Pf}' A_n + (\text{sub-leading})\,. \tag{32}$$

## 3 GALILEON, DIRAC–BORN–INFELD, EINSTEIN–MAXWELL–SCALAR

In this section we discuss the first class of the double soft theorems, which apply to scalar amplitudes in the special Galileon (sGal), Dirac–Born–Infeld (DBI), and Einstein–Maxwell–Scalar (EMS). The integrand $I_N$ for these three cases can be unified as

$$I_N = (\text{Pf}X_N)^{1-m} \, (\text{Pf}' A_N)^{3+m}\,, \tag{33}$$

where $m = 1, 0, -1$ corresponds to sGal, DBI and EMS respectively.

As commented in Subsection 2.1 there could be poles at $\rho = \infty$ when we deform the contour. But we are going to ignore any possible existence of this pole until the end of the discussion.

At the leading order, by (21) we have

$$\begin{aligned} S^{*(0)} &= \oint \frac{d\rho}{2\pi i} \, \frac{1}{\sum_{b=1}^{n} \frac{k_b \cdot (p+q)}{\rho - \sigma_b}} \, \frac{-\,\xi_1^2}{\tau\, p \cdot q} \left( \frac{-1}{\tau\, \xi_1} \right)^{1-m} \left( \frac{\tau\, p \cdot q}{-\xi_1} \right)^{3+m} \\ &= -\frac{\tau\, (\tau^2 \, p \cdot q)^m}{4} \oint \frac{d\rho}{2\pi i} \, \frac{\left( \sum_{b=1}^{n} \frac{k_b \cdot (p-q)}{\rho - \sigma_b} \right)^2}{\sum_{b=1}^{n} \frac{k_b \cdot (p+q)}{\rho - \sigma_b}}\,. \end{aligned} \tag{34}$$

The contour deformation picks up a simple pole at $\rho = \sigma_a$ for every $a \in \{1, \ldots, n\}$, and we obtain

$$S^{*(0)} = \frac{\tau\, (\tau^2 \, p \cdot q)^m}{4} \, \sum_{a=1}^{n} \frac{(k_a \cdot (p-q))^2}{k_a \cdot (p+q)}\,. \tag{35}$$

This completes the discussion at the leading order. Next we move on to the sub-leading order.

### 3.1 $S^{(1)}$ from Matching Delta Constraints

At the sub-leading order, we first handle the contribution from expanding the higher-point delta constraints. It is sufficient just to focus on the terms corresponding to any chosen label (say, label $a$) in (18), i.e.,

$$\begin{aligned}&\int \frac{d\mu_n}{\delta(g_a)}\,\delta'(g_a)\,I_n \oint \frac{d\rho}{2\pi i}\,\frac{\tau\,k_a\cdot(p+q)}{\sigma_a-\rho}\,\frac{1}{\sum_{b=1}^{n}\frac{k_b\cdot(p+q)}{\rho-\sigma_b}}\,\frac{-\xi_1^2}{\tau\,p\cdot q}\left(\frac{-1}{\tau\,\xi_1}\right)^{1-m}\left(\frac{\tau\,p\cdot q}{-\xi_1}\right)^{3+m}\\ &=\int \frac{d\mu_n}{\delta(g_a)}\,\delta'(g_a)\,I_n\,\frac{-\tau^2\,(\tau^2\,p\cdot q)^m}{4}\oint \frac{d\rho}{2\pi i}\,\frac{k_a\cdot(p+q)}{\sigma_a-\rho}\,\frac{\left(\sum_{b=1}^{n}\frac{k_b\cdot(p-q)}{\rho-\sigma_b}\right)^2}{\sum_{b=1}^{n}\frac{k_b\cdot(p+q)}{\rho-\sigma_b}}\,.\end{aligned} \tag{36}$$

By contour deformation we encounter a double pole at $\rho=\sigma_a$ and a simple pole at $\rho=\sigma_b$ for every other $b\in\{1,\ldots,n\}$. This computation leads to

$$\int \frac{d\mu_n}{\delta(g_a)}\,\delta'(g_a)\,I_n \sum_{\substack{b=1\\ b\neq a}}^{n}\frac{\tau^2\,(\tau^2\,p\cdot q)^m}{\sigma_a-\sigma_b}\,\frac{(k_a\cdot p\;k_b\cdot q-k_a\cdot q\;k_b\cdot p)^2}{k_a\cdot(p+q)\;k_b\cdot(p+q)}\,. \tag{37}$$

This result can be interpreted as coming from the action of an operator $S^{(1)}$ on one of the delta constraints in the corresponding lower-point amplitude, i.e.,

$$\int \frac{d\mu_n}{\delta(g_a)}\,S^{(1)}[\delta(g_a)]\,I_n\,, \tag{38}$$

where

$$S^{(1)} := \frac{\tau^2\,(\tau^2\,p\cdot q)^m}{2}\sum_{b=1}^{n}\frac{k_b\cdot(p-q)}{k_b\cdot(p+q)}\,p_\mu\,q_\nu\,J_b^{\mu\nu}\,,\qquad J_b^{\mu\nu}:=k_b^\mu\frac{\partial}{\partial k_{b\,\nu}}-k_b^\nu\frac{\partial}{\partial k_{b\,\mu}}\,. \tag{39}$$

This means the expansion of delta constraints on the higher-point side exactly matches the $S^{(1)}$ acting on delta constraints on the lower-point side.

### 3.2 Remaining Terms from the Expansion

We go on to study the remaining sub-leading terms from the expansion of $M_N$. These includes the sub-leading contributions from (17), ($1-m$ copies of) (28), and ($3+m$ copies of) (29).

Let us first focus on the second part of (29), which leads to the following contribution to the expansion of $M_N$ at the sub-leading order

$$\begin{aligned}
&\int d\mu_n \, \frac{(3+m)\, I_n}{\mathrm{Pf}' A_n} \oint \frac{d\rho}{2\pi i} \, \frac{1}{\sum_{c=1}^{n} \frac{k_c\cdot(p+q)}{\rho-\sigma_c}} \, \frac{-\xi_1^2}{\tau\, p\cdot q} \left(\frac{-1}{\tau\,\xi_1}\right)^{1-m} \left(\frac{\tau\, p\cdot q}{-\xi_1}\right)^{2+m} \\
&\qquad\qquad \times \sum_{3\le a<b\le n} \frac{\tau^2 \,(k_a\cdot p\, k_b\cdot q - k_a\cdot q\, k_b\cdot p)}{(\rho-\sigma_a)\,(\rho-\sigma_b)} \, \frac{(-1)^{a+b+1}\,\mathrm{Pf}[A_n]_{\hat{1},\hat{2},\hat{a},\hat{b}}}{(\sigma_1-\sigma_2)} \\
&= \int d\mu_n \, \frac{(3+m)\, I_n}{\mathrm{Pf}' A_n} \, \frac{\tau^2\,(\tau^2\, p\cdot q)^m}{2} \\
&\qquad \times \oint \frac{d\rho}{2\pi i} \, \frac{\sum_{c=1}^{n} \frac{k_c\cdot(p-q)}{\rho-\sigma_c}}{\sum_{c=1}^{n} \frac{k_c\cdot(p+q)}{\rho-\sigma_c}} \sum_{3\le a<b\le n} \frac{(k_a\cdot p\, k_b\cdot q - k_a\cdot q\, k_b\cdot p)}{(\rho-\sigma_a)\,(\rho-\sigma_b)} \, \frac{(-1)^{a+b+1}\,\mathrm{Pf}[A_n]_{\hat{1},\hat{2},\hat{a},\hat{b}}}{(\sigma_1-\sigma_2)} \, .
\end{aligned} \tag{40}$$

Due to the presence of sgn in the summation $\sum_{a,b=3}^{n}$ the term with $a=b$ is zero, and so contour deformation picks up a simple pole at $\rho=\sigma_a \;\forall a\in\{1,\ldots,n\}$, which results in

$$\int d\mu_n \, \frac{(3+m)\, I_n}{\mathrm{Pf}' A_n} \, \frac{-\tau^2\,(\tau^2\, p\cdot q)^m}{2} \sum_{a=3}^{n} \frac{k_a\cdot(p-q)}{k_a\cdot(p+q)} \sum_{\substack{b=3\\ b\neq a}}^{n} \frac{\theta_{a,b}\,(k_a\cdot p\, k_b\cdot q - k_a\cdot q\, k_b\cdot p)}{\sigma_a-\sigma_b} \, \frac{(-1)^{a+b+1}\,\mathrm{Pf}[A_n]_{\hat{1},\hat{2},\hat{a},\hat{b}}}{(\sigma_1-\sigma_2)} \, , \tag{41}$$

where $\theta_{a,b} := +1$ if $a<b$ and $\theta_{a,b} := -1$ if $a>b$. One can check that this is identical to

$$\int d\mu_n \, S^{(1)}[I_n] = \int d\mu_n \, (\mathrm{Pf}X_n)^{1-m}\, S^{(1)}\big[(\mathrm{Pf}' A_n)^{3+m}\big] \, , \tag{42}$$

with $S^{(1)}$ the same as that defined in (39). At this point, we have studied all the sub-leading terms on the higher-point side that account for the entire $S^{(1)}[M_n]$ on the lower-point side. But we see even more terms on the higher-point side. Do these remaining terms all cancel way?

To answer this question we collect all the other remaining sub-leading terms in the expansion

$$\begin{aligned}
&\int d\mu_n \, I_n \oint \frac{d\rho}{2\pi i} \, \frac{1}{\sum_{c=1}^{n} \frac{k_c\cdot(p+q)}{\rho-\sigma_c}} \, \frac{-\xi_1^2}{\tau\, p\cdot q} \left(\frac{-1}{\tau\,\xi_1}\right)^{1-m} \left(\frac{\tau\, p\cdot q}{-\xi_1}\right)^{3+m} \\
&\qquad \left( \underbrace{-\frac{\tau\,\xi_1}{2} \, \frac{\sum_{b=1}^{n} \frac{k_b\cdot(p-q)}{(\rho-\sigma_b)^2}}{\sum_{b=1}^{n} \frac{k_b\cdot(p+q)}{\rho-\sigma_b}} + 3\,\tau\, \frac{\xi_2}{\xi_1}}_{d\mu_N} \underbrace{-(1-m)\, \frac{\tau\,\xi_2}{\xi_1}}_{\mathrm{Pf}X_N} \underbrace{-(3+m)\, \frac{\tau\,\xi_2}{\xi_1}}_{\mathrm{Pf}' A_N} \right) \\
&= \int d\mu_n \, I_n \, \frac{-\tau^2\,(\tau^2\, p\cdot q)^m}{4} \oint \frac{d\rho}{2\pi i} \, \frac{\sum_{b=1}^{n} \frac{k_b\cdot(p-q)}{\rho-\sigma_b}}{\sum_{b=1}^{n} \frac{k_b\cdot(p+q)}{\rho-\sigma_b}} \left( -\frac{\sum_{b=1}^{n} \frac{k_b\cdot(p-q)}{(\rho-\sigma_b)^2}}{\sum_{b=1}^{n} \frac{k_b\cdot(p+q)}{\rho-\sigma_b}} + \frac{\sum_{b=1}^{n} \frac{k_b\cdot(p+q)}{(\rho-\sigma_b)^2}}{\sum_{b=1}^{n} \frac{k_b\cdot(p-q)}{\rho-\sigma_b}} \right) ,
\end{aligned} \tag{43}$$

where in the second line we remind the reader of the origin of each term. A careful study shows that this again has a simple pole at $\rho = \sigma_a \ \forall a \in \{1, \ldots, n\}$ (instead of a double pole), and the result is

$$\frac{\tau^2 (\tau^2 \, p \cdot q)^m}{4} \sum_{a=1}^{n} \left( 1 - \frac{(k_a \cdot (p - q))^2}{(k_a \cdot (p + q))^2} \right) M_n \,. \tag{44}$$

This might look a bit puzzling, because it seems to belong to $S^{(0)}$ but at the sub-leading order instead. Nevertheless, let us complete the analysis by studying the behavior of the formula at $\rho = \infty$.

### 3.3 *Pole at Infinity*

To study the behavior at infinity, we directly apply formulas (23), (31) and (32), which results in

$$(M_N)_\infty = \int d\mu_n \, I_n \oint \frac{d\rho}{2\pi i} \left( \frac{-4\,\rho^3}{\tau^4 \, (p \cdot q)^2} \, \frac{2\,(\tau^2 \, p \cdot q)^{3+m}}{(2\,i\,\rho)^4} + \mathcal{O}(\rho^{-2}) \right) . \tag{45}$$

Obviously there is a simple pole at infinity due to the leading term, and we need to subtract the residue associated to it, which gives rise to the contribution

$$-\frac{(\tau^2 \, p \cdot q)^{1+m}}{2} \, M_n \,. \tag{46}$$

Note that since all the sub-leading terms in the integrand in (45) are suppressed by $\rho^{-2}$, the result (46) is exact to all orders in $\tau$.

### 3.4 *Double Soft Theorem Up to Sub-Leading Order*

From the expansion of the higher-point side $M_N$, we have obtained (35) at the leading order in $\tau$, and (37), (41), (44), (46) at the sub-leading order. Among these, (35), (37) and (41) together can be re-written as $(S^{*(0)} + S^{(1)})[M_n]$. We are still left with (44) and (46), both of which are just a factor times the lower-point amplitude.

We can deform $S^{*(0)}$ at the sub-leading order and define the following operator

$$\frac{\tau \, (\tau^2 \, p \cdot q)^m}{4} \sum_{a=1}^{n} \left( \frac{(k_a \cdot (p - q))^2}{k_a \cdot (p + q) + \tau \, p \cdot q} + \tau \, p \cdot q \right) - \frac{(\tau^2 \, p \cdot q)^{1+m}}{2} \,. \tag{47}$$

One can check that (44) and (46) together are identical to the expansion of (47) at sub-leading order. To make things nicer, we apply momentum conservation to merge the last term of (47) into the summation as well, without showing any explicit dependence of the summand on $n$. Also, observe that both $S^{(0)}$ and $S^{(1)}$ contain an overall factor $(\tau^2 \, p \cdot q)^m$. If we bring it out then both operators are completely independent of $m$ and thus universal to this class of theories.

As a result of doing these re-arrangement, let us re-define the two operators to be

$$S^{(0)} := \frac{\tau}{4} \sum_{a=1}^{n} \left( \frac{(k_a \cdot (p-q))^2}{k_a \cdot (p+q) + \tau\, p \cdot q} + k_a \cdot p + k_a \cdot q + \tau\, p \cdot q \right), \tag{48}$$

$$S^{(1)} := \frac{\tau^2}{2} \sum_{b=1}^{n} \frac{k_b \cdot (p-q)}{k_b \cdot (p+q)}\, p_\mu\, q_\nu\, J_b^{\mu\nu}. \tag{49}$$

Then up to sub-leading order we have

$$M_N = (\tau^2\, p \cdot q)^m\, (S^{(0)} + S^{(1)})[M_n] + \mathcal{O}(\tau^{3+2m}). \tag{50}$$

## 4 NON-LINEAR SIGMA MODEL, YANG–MILLS–SCALAR

In this section we discuss the second class of double soft theorems, which apply to scalar amplitudes in the $U(N)$ non-linear sigma model (NLSM) and Yang–Mills–Scalar (YMS). For these theories we consider partial amplitude instead of the full amplitude, and without loss of generality we consider the canonical ordering (let us abbreviat it as $\alpha_I$). The integrand $I_N$ for these two cases can be unified as

$$I_N(\alpha) = C_N(\alpha)\, (\mathrm{Pf} X_N)^{-m}\, (\mathrm{Pf}' A_N)^{2+m}, \tag{51}$$

where $m = 0, -1$ corresponds to NLSM and YMS respectively.

First of all we have a quick check of the behavior at $\rho = \infty$. By (23), (30), (31) and (32) we obtain

$$(M_N(\alpha_I))_\infty = \int d\mu_n\, I_n \oint \frac{d\rho}{2\pi i} \left( \frac{-4\,\rho^3}{\tau^4\,(p \cdot q)^2}\, \frac{(\sigma_n - \sigma_1)\,(\tau^2\, p \cdot q)^{2+m}}{(2\,i\,\rho)^4\,\rho^2} + \mathcal{O}(\rho^{-4}) \right). \tag{52}$$

Hence for this class the formula is completely regular at infinity and there is no worry about this region at all.

By (21) we have

$$\begin{aligned} S^{*(0)} &= \oint \frac{d\rho}{2\pi i} \frac{1}{\sum_{b=1}^{n} \frac{k_b \cdot (p+q)}{\rho - \sigma_b}}\, \frac{-\xi_1^2}{\tau\, p \cdot q}\, \frac{(\sigma_n - \sigma_1)}{\tau\,(\sigma_n - \rho)\,(-\xi_1)\,(\rho - \sigma_1)} \left( \frac{-1}{\tau\,\xi_1} \right)^{-m} \left( \frac{\tau\, p \cdot q}{-\xi_1} \right)^{2+m} \\ &= \frac{(\tau^2\, p \cdot q)^m}{2} \oint \frac{d\rho}{2\pi i} \frac{\sum_{b=1}^{n} \frac{k_b \cdot (p-q)}{\rho - \sigma_b}}{\sum_{b=1}^{n} \frac{k_b \cdot (p+q)}{\rho - \sigma_b}}\, \frac{(\sigma_n - \sigma_1)}{(\sigma_n - \rho)\,(\rho - \sigma_1)}. \end{aligned} \tag{53}$$

Contour deformation picks up a simple pole at $\rho = \sigma_n$ and another at $\rho = \sigma_1$, we so we obtain

$$S^{*(0)} = \frac{(\tau^2\, p \cdot q)^m}{2} \left( \frac{k_n \cdot (p-q)}{k_n \cdot (p+q)} - \frac{k_1 \cdot (p-q)}{k_1 \cdot (p+q)} \right). \tag{54}$$

This completes the discussion at the leading order.

### 4.1 $S^{(1)}$ from Matching Delta Constraints

Similar to the discussion in the previous section, at the sub-leading order we first have a look at the expansion of the delta constraints in on the higher-point side. For each delta constraint labeled by $a \in \{2, \ldots, n-1\}$, its corresponding contribution is

$$\int \frac{d\mu_n}{\delta(g_a)}\, \delta'(g_a)\, I_n(\alpha_I) \oint \frac{d\rho}{2\pi i}\, \frac{\tau\, k_a \cdot (p+q)}{\sigma_a - \rho}\, \frac{1}{\sum_{b=1}^{n} \frac{k_b\cdot(p+q)}{\rho-\sigma_b}}\, \frac{-\xi_1^2}{\tau\, p\cdot q}$$
$$\frac{(\sigma_n - \sigma_1)}{\tau\,(\sigma_n-\rho)\,(-\xi_1)\,(\rho-\sigma_1)} \left(\frac{-1}{\tau\,\xi_1}\right)^{-m} \left(\frac{\tau\, p\cdot q}{-\xi_1}\right)^{2+m} \tag{55}$$
$$= \int \frac{d\mu_n}{\delta(g_a)}\, \delta'(g_a)\, I_n(\alpha_I)\, \frac{\tau\,(\tau^2\, p\cdot q)^m}{2} \oint \frac{d\rho}{2\pi i}\, \frac{k_a\cdot(p+q)}{\sigma_a-\rho}\, \frac{\sum_{b=1}^{n} \frac{k_b\cdot(p-q)}{\rho-\sigma_b}}{\sum_{b=1}^{n} \frac{k_b\cdot(p+q)}{\rho-\sigma_b}}\, \frac{(\sigma_n-\sigma_1)}{(\sigma_n-\rho)\,(\rho-\sigma_1)}\,.$$

Since $a \neq n, 1$, contour deformation picks up three simple poles, at $\sigma_a, \sigma_n, \sigma_1$ respectively, and the result is

$$-\int \frac{d\mu_n}{\delta(g_a)}\, \delta'(g_a)\, I_n(\alpha_I) \left(\frac{\tau\,(\tau^2\, p\cdot q)^m}{\sigma_a-\sigma_n}\, \frac{k_a\cdot p\; k_n\cdot q - k_a\cdot q\; k_n\cdot p}{k_n\cdot(p+q)} - (n \leftrightarrow 1)\right). \tag{56}$$

As is expected, this is equivalent to

$$\int \frac{d\mu_n}{\delta(g_a)}\, S^{(1)}[\delta(g_a)]\, I_n(\alpha_I)\,, \tag{57}$$

where

$$S^{(1)} := \tau\,(\tau^2\, p\cdot q)^m \left(\frac{p_\mu\, q_\nu}{k_n\cdot(p+q)}\, J_n^{\mu\nu} - \frac{p_\mu\, q_\nu}{k_1\cdot(p+q)}\, J_1^{\mu\nu}\right). \tag{58}$$

In the measure $d\mu_N$ of the higher-point side, we may as well have $\delta(f_a)$ for $a = n$ or $a = 1$. In these two cases after the contour deformation the pole at $\rho = \sigma_n$ or $\rho = \sigma_1$ (respectively) is a double pole, but as one can check, the final result obtained is again equivalent to (57). We do not bother to present this calculation explicitly here.

### 4.2 Remaining Terms from the Expansion

We go on to study the remaining sub-leading terms from the expansion of $M_N$. These includes the sub-leading contributions from (17), (24), ($-m$ copies of) (28), and ($2+m$ copies of) (29).

Let us again first focus on the second part of (29), which leads to the following contribution to the expansion of $M_N$ at the sub-leading order

$$\begin{aligned}
&\int d\mu_n \, \frac{(2+m)\, I_n}{\mathrm{Pf}' A_n} \oint \frac{d\rho}{2\pi i} \frac{1}{\sum_{c=1}^{n} \frac{k_c \cdot (p+q)}{\rho - \sigma_c}} \frac{-\xi_1^2}{\tau\, p \cdot q} \left(\frac{-1}{\tau\, \xi_1}\right)^{-m} \left(\frac{\tau\, p \cdot q}{-\xi_1}\right)^{1+m} \\
&\quad \times \frac{(\sigma_n - \sigma_1)}{\tau\, (\sigma_n - \rho)\, (-\xi_1)\, (\rho - \sigma_1)} \sum_{3 \le a < b \le n} \frac{\tau^2\, (k_a \cdot p\, k_b \cdot q - k_a \cdot q\, k_b \cdot p)}{(\rho - \sigma_a)\, (\rho - \sigma_b)} \frac{(-1)^{a+b+1}\, \mathrm{Pf}[A_n]_{\hat{1},\hat{2},\hat{a},\hat{b}}}{(\sigma_1 - \sigma_2)} \\
&= \int d\mu_n \, \frac{(2+m)\, I_n}{\mathrm{Pf}' A_n} \left(-\tau\, (\tau^2\, p \cdot q)^m\right) \oint \frac{d\rho}{2\pi i} \frac{1}{\sum_{c=1}^{n} \frac{k_c \cdot (p+q)}{\rho - \sigma_c}} \\
&\quad \times \frac{(\sigma_n - \sigma_1)}{(\sigma_n - \rho)\, (\rho - \sigma_1)} \sum_{3 \le a < b \le n} \frac{(k_a \cdot p\, k_b \cdot q - k_a \cdot q\, k_b \cdot p)}{(\rho - \sigma_a)\, (\rho - \sigma_b)} \frac{(-1)^{a+b+1}\, \mathrm{Pf}[A_n]_{\hat{1},\hat{2},\hat{a},\hat{b}}}{(\sigma_1 - \sigma_2)} \,.
\end{aligned} \tag{59}$$

After contour deformation, we only encounter a simple pole a $\rho = \sigma_n$ and a simple pole at $\rho = \sigma_1$. As expected the result is identical to

$$\int d\mu_n \, S^{(1)}[I_n(\alpha_I)] = \int d\mu_n \, C_n(\alpha_I)\, (\mathrm{Pf} X_n)^{-m}\, S^{(1)}\big[(\mathrm{Pf}' A_n)^{2+m}\big] \,. \tag{60}$$

Similar to that in the first class, the remaining task is to check the result of the remaining terms at the sub-leading order in the expansion, which are

$$\begin{aligned}
&\int d\mu_n \, I_n(\alpha_I) \oint \frac{d\rho}{2\pi i} \frac{1}{\sum_{c=1}^{n} \frac{k_c \cdot (p+q)}{\rho - \sigma_c}} \frac{-\xi_1^2}{\tau\, p \cdot q} \frac{(\sigma_n - \sigma_1)}{\tau\, (\sigma_n - \rho)\, (-\xi_1)\, (\rho - \sigma_1)} \left(\frac{-1}{\tau\, \xi_1}\right)^{-m} \left(\frac{\tau\, p \cdot q}{-\xi_1}\right)^{2+m} \\
&\times \left( \underbrace{-\frac{\tau\, \xi_1}{2} \frac{\sum_{b=1}^{n} \frac{k_b \cdot (p-q)}{(\rho - \sigma_b)^2}}{\sum_{b=1}^{n} \frac{k_b \cdot (p+q)}{\rho - \sigma_b}} + 3\, \tau\, \frac{\xi_2}{\xi_1}}_{d\mu_N} \underbrace{- \frac{\tau\, \xi_1}{2\, (\sigma_n - \rho)} - \frac{\tau\, \xi_1}{2\, (\rho - \sigma_1)} - \frac{\tau\, \xi_2}{\xi_1}}_{C_N(\alpha_I)} \underbrace{- m\, \frac{\tau\, \xi_2}{\xi_1}}_{\mathrm{Pf} X_N} \underbrace{- (2-m)\, \frac{\tau\, \xi_2}{\xi_1}}_{\mathrm{Pf}' A_N} \right) \\
&= \int d\mu_n \, I_n(\alpha_I)\, \frac{(\tau^2\, p \cdot q)^{1+m}}{2\, \tau} \oint \frac{d\rho}{2\pi i} \frac{1}{\sum_{b=1}^{n} \frac{k_b \cdot (p+q)}{\rho - \sigma_b}} \frac{(\sigma_n - \sigma_1)}{(\sigma_n - \rho)\, (\rho - \sigma_1)} \left( - \frac{\sum_{b=1}^{n} \frac{k_b \cdot (p-q)}{(\rho - \sigma_b)^2}}{\sum_{b=1}^{n} \frac{k_b \cdot (p+q)}{\rho - \sigma_b}} - \frac{1}{\sigma_n - \rho} - \frac{1}{\rho - \sigma_1} \right) .
\end{aligned} \tag{61}$$

Contour deformation only picks up a simple pole at $\rho = \sigma_n$ and at $\rho = \sigma_1$ respectively, and this results in

$$\tau\, (\tau^2\, p \cdot q) \left( \frac{k_n \cdot q\; p \cdot q}{(k_n \cdot (p+1))^2} + \frac{k_1 \cdot p\; p \cdot q}{(k_1 \cdot (p+1))^2} \right) M_n(\alpha_I). \tag{62}$$

Since there is no contribution from $\rho=\infty$, from the expansion of the higher-point side $M_N(\alpha_I)$ we obtained (54) at the leading order in $\tau$, and (56), (59), (62) at the sub-leading order. Among these (54), (56) and (59) together can be re-written as $(S^{*(0)}+S^{(1)})[M_n(\alpha_I)]$. Similar to that in the first class, we can regard (62) as coming from the expansion of a modification of $S^{*(0)}$, which is

$$\frac{(\tau^2\,p\cdot q)^m}{2}\left(\frac{k_n\cdot(p-q)+\tau\,p\cdot q}{k_n\cdot p+k_n\cdot q+\tau\,p\cdot q}-\frac{k_1\cdot(p-q)-\tau\,p\cdot q}{k_1\cdot p+k_1\cdot q+\tau\,p\cdot q}\right). \tag{63}$$

Again, we see the factor $(\tau^2\,p\cdot q)^m$ is overall, and so we can bring it out and re-define the two operators so that they are universal to the second class of theories, as follows

$$S^{(0)}:=\frac{1}{2}\left(\frac{k_n\cdot(p-q)+\tau\,p\cdot q}{k_n\cdot p+k_n\cdot q+\tau\,p\cdot q}-\frac{k_1\cdot(p-q)-\tau\,p\cdot q}{k_1\cdot p+k_1\cdot q+\tau\,p\cdot q}\right), \tag{64}$$

$$S^{(1)}:=\tau\left(\frac{p_\mu\,q_\nu}{k_n\cdot(p+q)}\,J_n^{\mu\nu}-\frac{p_\mu\,q_\nu}{k_1\cdot(p+q)}\,J_1^{\mu\nu}\right). \tag{65}$$

With these, up to sub-leading order we have

$$M_N(\alpha_I)=(\tau^2\,p\cdot q)^m\,(S^{(0)}+S^{(1)})[M_n(\alpha_I)]+\mathcal{O}(\tau^{1+2m}). \tag{66}$$

## 5 MORE GENERAL CASES

So far all what we have studied are scalar amplitudes only, even when the theory may involve other types of particles. In this section, we extend the discussion in two directions. Firstly we are going to show that the scalar double soft theorem (50) is still valid in DBI when external photons are present (the only modification, as usual, is to regard $J_a^{\mu\nu}$ as the full angular momentum operator, i.e., including both the orbital and the spin part). And secondly, we are going to show that there is also a similar leading-order theorem for simultaneous emission of two soft photons in photon amplitudes in Born–Infeld (BI) and Einstein–Maxwell (EM).

Before the discussion, we yet need to introduce another building block in $I_N$ which accounts for vector (and in general tensor) particles. We define a $2N\times 2N$ anti-symmetric matrix $\Psi_N$, which has the following block structure

$$\Psi_N:=\left(\begin{array}{c|c} A_N & -C_N^T \\ \hline C_N & B_N\end{array}\right), \tag{67}$$

where $A_N$ is exactly the same $A_N$ as defined in (5), and we also have

$$(C_N)_{ab}:=\begin{cases}\frac{\epsilon_a\cdot k_b}{\sigma_a-\sigma_b}\,, & a\neq b\\ -\sum_{c\neq a}(C_N)_{ac}\,, & a=b\end{cases},\quad (B_N)_{ab}:=\begin{cases}\frac{\epsilon_a\cdot\epsilon_b}{\sigma_a-\sigma_b}\,, & a\neq b\\ 0\,, & a=b\end{cases}. \tag{68}$$

It is convenient to separate the range of the indices into two blocks as well, i.e., $\{1,\ldots,N:1,\ldots,N\}$. We also use the notation that, e.g., $[\Psi_N]_{\hat{1},\hat{2}:\hat{3}}$ denotes the minor of $\Psi_N$ obtained by deleting the rows and columns labeled by 1, 2 in the first block as well as the row and column labeled by 3 in the second block. Again matrix $\Psi$ generically has corank 2 on the support of the scattering equations, and so natural invariant quantities are $\mathrm{Pf}'[\Psi_N]_{:\hat{U}}:=\frac{(-1)^{a+b}}{\sigma_{ab}}\mathrm{Pf}'[\Psi_N]_{\hat{a},\hat{b}:\hat{U}}$ for some given label set $U$ (can be empty).

In a DBI amplitude let us denote the label set for scalars as $\mathfrak{s}$ and that for photons as $\gamma$. Then the integrand for a general DBI amplitude as well as those for photon amplitudes in BI and EM are

$$\begin{aligned} I_N^{\mathrm{DBI}} &:= \mathrm{Pf}[X_N]_{\hat{\gamma}}\, \mathrm{Pf}'[\Psi_N]_{:\hat{\mathfrak{s}}}\, (\mathrm{Pf}' A_N)^2 \,, \\ I_N^{\mathrm{BI}} &:= \mathrm{Pf}'\Psi_N\, (\mathrm{Pf}' A_N)^2 \,, \quad I_N^{\mathrm{EM}} := \mathrm{Pf} X_N\, \mathrm{Pf}' A_N\, \mathrm{Pf}'\Psi_N \,. \end{aligned} \tag{69}$$

## 5.1 *DBI Amplitudes with Mixed External States*

For a DBI amplitudes with both scalar and photon external states, with two soft scalars the soft theorem is still in the form of (50). However, as is usually the case, the $J_a^{\mu\nu}$ in $S^{(1)}$ is now the full angular momentum operator, containing both the orbital and the spin part $J_a^{\mu\nu} := J_{a,\mathrm{orbital}}^{\mu\nu} + J_{a,\mathrm{spin}}^{\mu\nu}$, with

$$J_{a,\mathrm{orbital}}^{\mu\nu} := k_a^{\mu} \frac{\partial}{\partial k_{a,\nu}} - k_a^{\nu} \frac{\partial}{\partial k_{a,\mu}} \,, \tag{70}$$

$$J_{a,\mathrm{spin}}^{\mu\nu}[\epsilon_b^{\rho}] := \delta_{ab}\, (\delta^{\nu\rho}\delta^{\mu}_{\sigma} - \delta^{\mu\rho}\delta^{\nu}_{\sigma})\epsilon_a^{\sigma} \,. \tag{71}$$

In this note we will only sketch the proof for the leading order in this case. The main task is to study the behavior of the new building block $\mathrm{Pf}'[\Psi_N]_{:\hat{\mathfrak{s}}}$ in the double soft limit. When evaluated on the degenerate solution the detailed structure of the matrix $[\Psi_N]_{:\hat{\mathfrak{s}}}$ is

$$[\Psi_N]_{:\hat{\mathfrak{s}}} \approx \left( \begin{array}{c|cc|c} A_{ab} & \frac{\tau\, k_a\cdot p}{\sigma_a-\rho} & \frac{\tau\, k_a\cdot q}{\sigma_a-\rho} & (-C^T)_{ad} \\ \hline \frac{\tau\, p\cdot k_b}{\rho-\sigma_b} & 0 & \frac{\tau\, p\cdot q}{-\xi_1} & \frac{\tau\, p\cdot \epsilon_d}{\rho-\sigma_d} \\ \frac{\tau\, q\cdot k_b}{\rho-\sigma_b} & \frac{\tau\, q\cdot p}{\xi_1} & 0 & \frac{\tau\, q\cdot \epsilon_d}{\rho-\sigma_d} \\ \hline C_{cb} & \frac{\tau\, \epsilon_c\cdot p}{\sigma_c-\rho} & \frac{\tau\, \epsilon_c\cdot q}{\sigma_c-\rho} & B_{cd} \end{array} \right) , \tag{72}$$

where $a, b \in \{1, \ldots, n\}$ and $c, d \in \gamma$ (we assume the last two labels denote scalars). Since the four blocks at the corner are completely finite we do not bother to write them explicitly. This is very similar to the case of matrix $A_N$, and we immediately know

$$\mathrm{Pf}'[\Psi_N]_{:\hat{\mathfrak{s}}} = \frac{\tau\, q\cdot p}{-\xi_1}\, \mathrm{Pf}'[\Psi_n]_{:\hat{\mathfrak{s}}} + \mathcal{O}(\tau^2) \,. \tag{73}$$

Recalling (29) we see that when we replace the integrand for scalar amplitude $\mathrm{Pf} X_N\, (\mathrm{Pf}' A_N)^3$ to that for mixed amplitude $\mathrm{Pf}[X_N]_{\hat{\gamma}}\, \mathrm{Pf}'[\Psi_N]_{:\hat{\mathfrak{s}}}\, (\mathrm{Pf}' A_N)^2$, the expression for $S^{*(0)}$ (34) stays the same, and hence we have exactly the same $S^{*(0)}$. Since $S^{*(0)}$ accounts for all the leading-order behavior, we conclude that we have the same double soft theorem at the leading order.

## 5.2 *Double Soft Photons Emission in Born–Infeld and Einstein–Maxwell*

We go on the investigate photon amplitudes in BI and EM, with two soft photons. We can use a single integrand for this class of amplitudes

$$I_N = (\mathrm{Pf} X_N)^{-m}\, (\mathrm{Pf}' A_N)^{2+m}\, \mathrm{Pf}'\Psi_N \,, \tag{74}$$

where $m = 0, -1$ denotes BI and EM respectively. The prescription of the limit is still the same as that for the soft scalars. Again the only thing we need the check is the behavior of the matrix $\Psi_N$ since now the second block of labels also includes $n+1$ and $n+2$.

To the order sufficient for our interest the structure of matrix $\Psi_N$ is

$$
\Psi_N \approx \left(\begin{array}{c:cc|c:cc}
A_{ab} & \frac{\tau\, k_a\cdot p}{\sigma_a-\rho} & \frac{\tau\, k_a\cdot q}{\sigma_a-\rho} & (-C_N^T)_{ad} & \frac{k_a\cdot\epsilon_{n+1}}{\sigma_a-\rho} & \frac{k_a\cdot\epsilon_{n+2}}{\sigma_a-\rho} \\
\hdashline
\frac{\tau\, p\cdot k_b}{\rho-\sigma_b} & 0 & \frac{\tau\, p\cdot q}{-\xi_1} & \frac{\tau\, p\cdot\epsilon_d}{\rho-\sigma_d} & (-C_N^T)_{n+1,n+1} & \frac{p\cdot\epsilon_{n+2}}{-\xi_1} \\
\frac{\tau\, q\cdot k_b}{\rho-\sigma_b} & \frac{\tau\, q\cdot p}{\xi_1} & 0 & \frac{\tau\, q\cdot\epsilon_d}{\rho-\sigma_d} & \frac{q\cdot\epsilon_{n+1}}{\xi_1} & (-C_N^T)_{n+2,n+2} \\
\hline
(C_N)_{cb} & \frac{\tau\,\epsilon_c\cdot p}{\sigma_c-\rho} & \frac{\tau\,\epsilon_c\cdot q}{\sigma_c-\rho} & B_{cd} & \frac{\epsilon_c\cdot\epsilon_{n+1}}{\sigma_c-\rho} & \frac{\epsilon_c\cdot\epsilon_{n+2}}{\sigma_c-\rho} \\
\hdashline
\frac{\epsilon_{n+1}\cdot k_a}{\rho-\sigma_a} & (C_N)_{n+1,n+1} & \frac{\epsilon_{n+1}\cdot q}{-\xi_1} & \frac{\epsilon_{n+1}\cdot\epsilon_d}{\rho-\sigma_d} & 0 & \frac{\epsilon_{n+1}\cdot\epsilon_{n+2}}{-\tau\,\xi_1} \\
\frac{\epsilon_{n+2}\cdot k_a}{\rho-\sigma_a} & \frac{\epsilon_{n+2}\cdot p}{\xi_1} & (C_N)_{n+2,n+2} & \frac{\epsilon_{n+2}\cdot\epsilon_d}{\rho-\sigma_d} & \frac{\epsilon_{n+2}\cdot\epsilon_{n+1}}{\tau\,\xi_1} & 0
\end{array}\right), \tag{75}
$$

where the two extra diagonal terms of matrix $C_N$ are approximated as

$$
(C_N)_{n+1,n+1} = \sum_{i=1}^{n} \frac{\epsilon_{n+1}\cdot k_i}{\rho-\sigma_i} - \frac{\epsilon_{n+1}\cdot q}{\xi_1} + \mathcal{O}(\tau) = \sum_{i=1}^{n} \frac{\epsilon_{n+1}\cdot p_i^{\perp}}{\rho-\sigma_i} + \mathcal{O}(\tau)\,, \tag{76}
$$

with $p_i^{\perp} := k_i - \frac{p\cdot k_i}{p\cdot q}\, q$ (We denote this new vector as $p_i^{\perp}$ because $p\cdot p_i^{\perp} = 0$). In the second equality above we applied the scattering equation labeled by $n+1$ to get rid of $\xi_1$. Similarly, we have

$$
(C_N)_{n+2,n+2} = \sum_{i=1}^{n} \frac{\epsilon_{n+2}\cdot q_i^{\perp}}{\rho-\sigma_i} + \mathcal{O}(\tau)\,, \tag{77}
$$

with $q_i^{\perp} := k_i - \frac{q\cdot k_i}{q\cdot p}\, p$. By studying carefully the scaling of each entry in (75) we observe that

$$
\mathrm{Pf}'\Psi_N = \mathrm{Pf}[\Psi_N]_{n+1,n+2:n+1,n+2}\, \mathrm{Pf}'\Psi_n + \mathcal{O}(\tau^2)\,, \tag{78}
$$

where $[\Psi_N]_{n+1,n+2:n+1,n+2}$ is the minor of $\Psi_N$ with entries $\{n+1, n+2 : n+1, n+2\}$ only. Plugging this intro the expression for $S^{*(0)}$ we obtain

$$
\begin{aligned}
S^{*(0)} &= \oint \frac{d\rho}{2\pi i}\, \frac{1}{\sum\limits_{b=1}^{n} \frac{k_b\cdot(p+q)}{\rho-\sigma_b}}\, \frac{-\xi_1^2}{\tau\, p\cdot q} \left(\frac{-1}{\tau\,\xi_1}\right)^{-m} \left(\frac{\tau\, p\cdot q}{-\xi_1}\right)^{2+m} \\
&\qquad\qquad \times \left( \frac{p\cdot q\;\epsilon_{n+1}\cdot\epsilon_{n+2}}{\xi_1^2} - \sum_{i,j=1}^{n} \frac{\epsilon_{n+1}\cdot p_i^{\perp}\;\epsilon_{n+2}\cdot q_i^{\perp}}{(\rho-\sigma_i)\,(\rho-\sigma_j)} - \frac{p\cdot\epsilon_{n+2}\; q\cdot\epsilon_{n+1}}{\xi_1^2} \right) \\
&= \oint \frac{d\rho}{2\pi i}\, \frac{1}{\sum\limits_{b=1}^{n} \frac{k_b\cdot(p+q)}{\rho-\sigma_b}}\, \frac{(\tau^2\, p\cdot q)^{1+m}}{-\tau} \left( \frac{p\cdot q\;\epsilon_{n+1}\cdot\epsilon_{n+2} - p\cdot\epsilon_{n+2}\; q\cdot\epsilon_{n+1}}{\xi_1^2} - \sum_{i,j=1}^{n} \frac{\epsilon_{n+1}\cdot p_i^{\perp}\;\epsilon_{n+2}\cdot q_i^{\perp}}{(\rho-\sigma_i)\,(\rho-\sigma_j)} \right).
\end{aligned} \tag{79}
$$

Recalling the solution of $\xi_1$ (15), although there seems to be a simple pole at $\rho = \infty$, as one can check its residue is proportional to

$$
\frac{p\cdot q\;\epsilon_{n+1}\cdot\epsilon_{n+2} - p\cdot\epsilon_{n+2}\; q\cdot\epsilon_{n+1}}{4\,(p\cdot q)^2} \left( \sum_{b=1}^{n} k_b\cdot(p-q) \right)^2 - \sum_{i,j=1}^{n} \epsilon_{n+1}\cdot p_i^{\perp}\;\epsilon_{n+2}\cdot q_i^{\perp}\,, \tag{80}
$$

which vanishes due to momentum conservation. Hence contour deformation again picks up a simple pole at every $\rho = \sigma_b$ with $b \in \{1, \ldots, n\}$, which leads to the result

$$\sum_{b=1}^{n} \frac{(\tau^2\, p \cdot q)^{1+m}}{\tau\, k_b \cdot (p+q)} \left( \frac{p \cdot q\, \epsilon_{n+1} \cdot \epsilon_{n+2} - p \cdot \epsilon_{n+2}\, q \cdot \epsilon_{n+1}}{4\,(p \cdot q)^2} (k_b \cdot (p-q))^2 - \epsilon_{n+1} \cdot p_b^{\perp}\, \epsilon_{n+2} \cdot q_b^{\perp} \right). \quad (81)$$

Since the original summation structure from the pfaffian is not at all changed by the $\rho$ integration, we observe that the soft factor can still be written as a pfaffian, and so

$$M_N = \frac{\tau^3\,(\tau^2\, p \cdot q)^{m-1}}{4} \sum_{a=1}^{n} \left( \frac{(k_b \cdot (p-q))^2}{k_b \cdot (p+q)} \mathrm{Pf}\mathcal{S}_b \right) M_n + \mathcal{O}(\tau^{4+2m})\,. \quad (82)$$

where $\mathcal{S}_b$ is a $4 \times 4$ anti-symmetric matrix

$$\mathcal{S}_b = \left( \begin{array}{cc|cc} 0 & p \cdot q & \hat{p}_b^{\perp} \cdot \epsilon_{n-1} & p \cdot \epsilon_{n+2} \\ -q \cdot p & 0 & -q \cdot \epsilon_{n+1} & \hat{q}_b^{\perp} \cdot \epsilon_{n+2} \\ \hline -\epsilon_{n+1} \cdot \hat{p}_b^{\perp} & \epsilon_{n+1} \cdot q & 0 & \epsilon_{n+1} \cdot \epsilon_{n+2} \\ -\epsilon_{n+2} \cdot p & -\epsilon_{n+2} \cdot \hat{q}_b^{\perp} & -\epsilon_{n+2} \cdot \epsilon_{n+1} & 0 \end{array} \right), \quad (83)$$

where $\hat{p}_b^{\perp} := \frac{2\, p \cdot q}{k_b \cdot (p-q)}\, p_b^{\perp}$ and $\hat{q}_b^{\perp} := \frac{2\, p \cdot q}{k_b \cdot (p-q)}\, q_b^{\perp}$.

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