talk at DAMTP, Cambridge Feb 03, 2015

Tree-Level S-Matrices
from Einstein to Yang-Mills, Born-Infeld and More

refs: 1409.8256, 1412.3479

* Review of the formulation
* compactify
* "squeeze"
* "compactify"

} three operations on the integrand

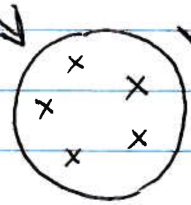
* Review

given a set of external states in a scattering process w/ momenta (& polarizations)

$$\{k, \epsilon\} \longmapsto \mathcal{M} = \underline{M(k, \epsilon)}\, \delta^D(\sum k)$$

└ though rational at tree level explicit expression are not unique
locality & unitarity usually hidden behind

scattering equations



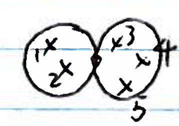
introduce an auxiliary punctured Riemann sphere to make unitarity & locality manifest

$\Downarrow$

Criteria: when approaching certain factorization channel the R.S. has to become degenerate

i.e. 5 points, $s_{12} \to 0$

$\Rightarrow$



for n particle scattering

parametrize the moduli space $\Sigma_{0,n}$ by $(\sigma_1, \cdots, \sigma_n)/SL(2,\mathbb{C})$

define : $\quad Q_a := \sum_{b\neq a,\, b=1}^{n} \frac{s_{ab}}{\sigma_a - \sigma_b}$

Scattering Equations : $\quad \boxed{Q_a = 0 \quad \forall a}$

linear dependence : $\sigma_a^m Q_a = 0, \quad m = 0,1,2$

$\Rightarrow \{\sigma_a\}$ fully localized for any given kinematics data

in total $(n-3)!$ solutions, counting multiplicity

(distinct for generic kinematics)

How do we use the scattering equations to construct amplitudes?

$$M = \int \underbrace{\frac{\prod_{a=1}^{n} d\sigma_a\, \delta(Q_a)}{(\text{redundancies})}}_{d\mu_n} I(k,\epsilon,\sigma) = \sum_{\text{solns.}}^{(n-3)!} \frac{I(k,\epsilon,\sigma)}{\text{Jacobian}(k,\sigma)}$$

(for explicit definition of the measure,

see our paper or my previous talks ; not important here)

An important constraint on I :

under $SL(2,\mathbb{C})$: $\quad \sigma_a \longmapsto \frac{\alpha\sigma_a + \beta}{\gamma\sigma_a + \delta} \quad \forall a, \quad \alpha\delta - \beta\gamma = 1$

$$I \longmapsto I \prod_{a=1}^{n} (\gamma\sigma_a + \delta)^2 (\gamma\sigma_a + \delta)^2$$

An intuitive example :

define : $\quad C(\alpha) := \frac{1}{\sigma_{\alpha(1),\alpha(2)}\, \sigma_{\alpha(2),\alpha(3)} \cdots \sigma_{\alpha(n),\alpha(1)}} \qquad (\sigma_{a,b} := \sigma_a - \sigma_b)$

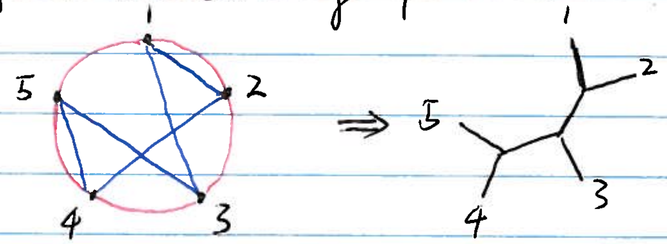
$I = C(\alpha)\, C(\beta)$ is a well-defined integrand

in general, scattering equations are hard to solve.

naively we would expect the result to be complicated

however $\quad \int d\mu_5\, C(12345)\, C(12453) = \frac{1}{s_{12}\, s_{45}}$.

a graphical understanding for this :



a diagram in massless ϕ^3 !

Similar graphical rule apply to ANY $C(\alpha)\, C(\beta)$.

Lesson : in principle, any tree amplitude in a theory of massless particles can be turned into an integration formula of this kind.

$$\mathcal{M} = \sum_{\substack{\text{Feynman} \\ \text{diagrams}}} \frac{N(k,\epsilon)}{s\, s \cdots s} = \int d\mu_n \left(\sum_{F.D.} N\, C\, C \right)$$

What's the use of this formulation ?

* Closed formulas from very simple building blocks
* Connection to (ambi-)twistor strings

→ our interest in this talk

Building blocks :

① $C_n := \sum_{\alpha \in S_{n-1}} \mathrm{tr}\left(T^{\alpha(1)} T^{\alpha(2)} \cdots T^{\alpha(n)}\right) C(\alpha)$

(for theories w/ non-trivial color/flavor groups)

② $\mathrm{Pf}'\Psi_{n:n} := \left(\mathrm{Pf}\,[\Psi]_{\underbrace{ab:n}}\right) \frac{(-1)^{a+b}}{\sigma_a - \sigma_b}$

└ rows/columns deleted

(for particles w/ spin > 0)

$\Psi_{n:n} = \left(\begin{array}{c|c} A & -C^T \\ \hline C & B \end{array}\right)$ an $2n \times 2n$ skew-symmetric matrix

A, B, C are $n \times n$

$$A_{ab} := \begin{cases} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} \\ 0 \end{cases} \quad B_{ab} := \begin{cases} \frac{\epsilon_a \cdot \epsilon_b}{\sigma_a - \sigma_b} \\ 0 \end{cases} \quad C_{a,b} := \begin{cases} \frac{\epsilon_a \cdot k_b}{\sigma_a - \sigma_b} & a \neq b \\ -\sum_{c \neq a} C_{ac} & a = b \end{cases}$$

* A special Galileon theory : $I = (\text{Pf}'A)^2 (\text{Pf}'A)^2$ (*)

A quick review of the most general Galileon :
a real scalar theory. invariant under
$$\delta\phi = a + b\cdot x$$

general Lagrangian : (assuming D dimensional spacetime)
$$\mathcal{L} = -\frac{1}{2}(\partial\phi\cdot\partial\phi) \sum_{m=0}^{\infty} \frac{g_m}{(D-m)!} \partial_{\mu_1}\partial^{\nu_1}\phi \cdots \partial_{\mu_m}\partial^{\nu_m}\phi \, \varepsilon^{\mu_1\cdots\mu_m\rho_{m+1}\cdots\rho_D}\varepsilon_{\nu_1\cdots\nu_m\rho_{m+1}\cdots\rho_D}$$
where g_m are coupling that can be choozen freely

The formula ~~above~~ (*) computes amplitudes in a special Galileon theory
where g_1, g_2 are free while
$g_m = g_m(g_1, g_2)$ $\forall m > 2$ some fixed function

How special is this ?
Z_2 invariant ;
enhanced symmetry under
$$\delta\phi = S_{\mu\nu}(x^\mu + \beta\partial^\mu\phi)(x^\nu + \beta\partial^\nu\phi) + S_{\mu\nu}\,\alpha\,\partial^\mu\phi\partial^\nu\phi$$

especially, if we set $g_1 = 0$, then (for some α)
$$g_m = \begin{cases} \frac{(-\alpha)^{\frac{m}{2}}}{(m+1)!} & \text{even } m \\ 0 & \text{odd } m \end{cases}$$

For more details about this special Galileon, see:
Hinterbichler, Joyce , arXiv: 1501.07600