Supplementary Document

Scattering of Massless Particles in Arbitrary Dimension: Soft Limits and Factorizations, Version II

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Abstract: This note supplements the recent paper [1] of the current authors by providing a detailed proof for both the soft limits and the factorizations of the formula proposed therein.

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We study in turn the soft limit and factorizations on physical poles of the formulas for Yang-Mills and gravity tree-level S-matrices in any dimensions,

$$\begin{aligned} A_n =& \frac{1}{\mathrm{vol}\,\mathrm{SL}(2,\mathbb{C})} \int \frac{d^n\sigma}{\sigma_{12}\cdots\sigma_{n1}} \prod_a{}' \delta(\sum_{b\neq a} \frac{s_{a,b}}{\sigma_{a,b}})\, \mathrm{Pf}'\Psi(k,\epsilon,\sigma). \\ M_n =& \frac{1}{\mathrm{vol}\,\mathrm{SL}(2,\mathbb{C})} \int d^n\sigma \prod_a{}' \delta(\sum_{b\neq a} \frac{s_{a,b}}{\sigma_{a,b}})\, \mathrm{Pf}'\Psi(k,\epsilon,\sigma)\, \mathrm{Pf}'\Psi(k,\tilde{\epsilon},\sigma), \end{aligned} \tag{0.1}$$

where $\sigma_{a,b} = \sigma_a - \sigma_b$ [1]. The $2n \times 2n$ matrix Ψ is defined as

$$\Psi_{a,b} = \begin{cases} \frac{s_{a,b}}{\sigma_{a,b}}, & a\neq b \\ 0, & a=b \end{cases}, \quad \Psi_{a+n,b} = \begin{cases} \frac{(\epsilon_a+k_b)^2}{\sigma_{a,b}}, & a\neq b \\ -\sum_{c\neq a} \frac{(\epsilon_a+k_c)^2}{\sigma_{a,c}}, & a=b \end{cases}, \quad \Psi_{a+n,b+n} = \begin{cases} \frac{(\epsilon_a+\epsilon_b)^2}{\sigma_{a,b}}, & a\neq b \\ 0, & a=b \end{cases}, \tag{0.2}$$

with a, b ranging from 1 to n, and the elements $\Psi_{a,b+n}$ are defined via the requirement of antisymmetry on Ψ. And the reduced Pfaffian $\mathrm{PF}'\Psi$ is defined as

$$\mathrm{Pf}'\Psi = \frac{(-1)^{i+j}}{\sigma_{i,j}} \mathrm{Pf}\Psi^{ij}_{ij}, \tag{0.3}$$

where the $2(n-1)\times 2(n-1)$ matrix Ψ^{ij}_{ij} is obtained from Ψ by deleting the i^{th} and j^{th} row and column, with $i, j \leq n$. Here the polarization vectors ϵ's are understood as null. To simplify notations, we also denote $\tilde{\Psi} = \Psi(k,\tilde{\epsilon},\sigma)$. Both formulas can be written in a form where all the integrals have been performed

$$A_n = \sum_{\{\sigma\}\in\text{ solutions}} \frac{1}{\sigma_{12}\cdots\sigma_{n1}} \frac{\mathrm{Pf}'\Psi(k,\epsilon,\sigma)}{\det'\Phi} \tag{0.4}$$

and

$$M_n = \sum_{\{\sigma\}\in\text{ solutions}} \frac{\mathrm{Pf}'\Psi(k,\epsilon,\sigma)\,\mathrm{Pf}'\Psi(k,\tilde{\epsilon},\sigma)}{\det'\Phi}. \tag{0.5}$$

A slightly different convention of the formula and the reduced Pfaffian as well as its consistency (i.e. gauge invariance and permutation invariance) have been discussed in detail in [1].

[1] Note that the formulas above differ from those in [1] by some overall constant factors that can be absorbed into the definition of the coupling constants. More explicitly, $\mathcal{M}_n^{\text{YM,here}} = \frac{1}{2}\mathcal{M}_n^{\text{YM,there}}$ and $\mathcal{M}_n^{\text{gravity,here}} = 2^{n-1}\mathcal{M}_n^{\text{gravity,there}}$. Formula with the current convention (which coincides with that in [2]) exactly match the standard results from the spin-helicity formalism when restricting to 4 dimensions. This convention is also nicer from the point of view of factorizations.

1 Soft Limit

As discussed in detail in [2], when we take the soft limit $k_n \to 0$, $k_n \equiv \varepsilon \hat{k}_n \to 0$, to leading order in ε, $n-1$ of the scattering equations become identical to those of a system with $n-1$ particles. The last equation

$$\sum_{b \neq n} \frac{s_{n,b}}{\sigma_n - \sigma_b} = 0 \tag{1.1}$$

becomes a polynomial for σ_n of degree $n-3$ (due to momentum conservation). With the choice that σ_n unfixed and the n^{th} equation included, the measure and delta functions split into the $(n-1)$-particle part and the part for the n^{th} particle,

$$\frac{d^{\,n}\sigma}{\mathrm{vol}\,\mathrm{SL}(2,\mathbb{C})} \prod_{a=1}^{n}{}' \delta(\sum_{b\neq a} \frac{s_{a,b}}{\sigma_{a,b}}) \to \frac{d^{\,n-1}\sigma}{\mathrm{vol}\,\mathrm{SL}(2,\mathbb{C})} \prod_{a=1}^{n-1}{}' \delta(\sum_{b\neq a} \frac{s_{a,b}}{\sigma_{a,b}}) d\sigma_n \delta(\sum_{a=1}^{n-1} \frac{s_{n,a}}{\sigma_{n,a}})\,. \tag{1.2}$$

The kinematic-dependence of the integrand of Yang-Mills/gravity amplitude is encoded in the reduced Pfaffian $\mathrm{Pf}'\Psi$, and only the n^{th} column and row of Ψ depend on k_n under consideration. The Pfaffian of a $2m \times 2m$ matrix E satisfies a recursion relation of the form (ignoring the overall sign)

$$\mathrm{Pf}(E) = \sum_{q=1}^{2m} (-1)^q e_{pq} \mathrm{Pf}(E^{pq}_{pq}). \tag{1.3}$$

Using this formula to expand $\mathrm{Pf}\Psi^{ij}_{ij}$ (assuming $i \neq n, j \neq n$) setting $p = n$, and note that Ψ_{na} for $a = 1, \ldots, 2n-1$ are proportional to k_n (except $\Psi_{nn} = 0$) while $\Psi_{n,2n} = \sum_{a=1}^{n-1} \frac{2\epsilon_n \cdot k_a}{\sigma_{n,a}}$, thus one finds that in the soft limit only one term contributes and gives

$$\mathrm{Pf}\Psi^{ij}_{ij} \to \sum_{a=1}^{n-1} \frac{2\epsilon_n \cdot k_a \prod_{b\neq a,n} \sigma_{n,b}}{\prod_{c=1}^{n-1} \sigma_{n,c}} \mathrm{Pf}\Psi^{ijn(2n)}_{ijn(2n)} = \sum_{a=1}^{n-1} \frac{2\epsilon_n \cdot k_a \prod_{b\neq a,n} \sigma_{n,b}}{\prod_{c=1}^{n-1} \sigma_{n,c}} \mathrm{Pf}\Psi_{n-1}{}^{ij}_{ij}, \tag{1.4}$$

where in the second equality the fact that $\mathrm{Pf}\Psi^{ijn(2n)}_{ijn(2n)}$ is independent of k_n and ϵ_n gives rise to $\mathrm{Pf}'\Psi_{n-1}$, i.e., the reduced Pfaffian for $n-1$ particles. For Yang-Mills amplitudes, using the explicit formula (0.4) in the soft limit one finds

$$A_n \to \sum_{i=1}^{(n-4)!} \oint_\Gamma d\sigma_n \frac{\sum_{a\neq n} \epsilon_n \cdot k_a \prod_{b\neq a,n} \sigma^{(i)}_{n,b}}{\sum_{a\neq n} k_n \cdot k_a \prod_{b\neq a,n} \sigma^{(i)}_{n,b}} \frac{\sigma^{(i)}_{n-1,1}}{\sigma^{(i)}_{n-1,n}\sigma^{(i)}_{n,1}} \frac{1}{\sigma^{(i)}_{1,2}\cdots\sigma^{(i)}_{n-1,1}} \frac{\mathrm{Pf}'\Psi^{(i)}_{n-1}}{\det'\Phi^{(i)}_{n-1}}, \tag{1.5}$$

where the sum is over $(n-4)!$ solutions of the equations for $n-1$ particles and all $\sigma^{(i)}_a$'s with $a \in \{1, 2, \ldots, n-1\}$ are taken to be evaluated on the i^{th} solution. Since σ_a's are taken to be complex numbers in our formulas, the delta functions imposing the scattering equations are in fact poles and all our integrals are contour integrals; here the contour Γ for the integral over σ_n is defined to encircle the $n-3$ zeroes of the first factor in the denominator. Similarly for gravity amplitudes using (0.5) one finds

$$M_n \to \sum_{i=1}^{(n-4)!} \oint_\Gamma d\sigma_n \frac{\sum_{a\neq n} \epsilon_n \cdot k_a \prod_{b\neq a,n} \sigma_{n,b}^{(i)}}{\sum_{a\neq n} k_n \cdot k_a \prod_{b\neq a,n} \sigma_{n,b}^{(i)}} \frac{\sum_{a\neq n} 2\epsilon_n \cdot k_a \prod_{b\neq a,n} \sigma_{n,b}^{(i)}}{\prod_{c=1}^{n-1} \sigma_{n,c}^{(i)}} \frac{\mathrm{Pf}'\Psi_{n-1}^{(i)} \mathrm{Pf}'\tilde{\Psi}_{n-1}^{(i)}}{\det'\Phi_{n-1}^{(i)}}. \tag{1.6}$$

Let us study the pole at infinity: as $\sigma_n \to \infty$, the first factor goes like a constant since the polynomial in the numerator is also of degree $n-3$, and in both cases the second factor goes like $1/\sigma_n^2$ (For gravity (1.6) the highest-order terms in the numerator of the second factor cancel out due to momentum conservation and $\epsilon_n \cdot k_n = 0$). Hence for both Yang-Mills and gravity there is no contribution at infinity, and using the residue theorem the contour can be deformed to enclose the poles of the second factor in both cases. For Yang-Mills, there are only two poles at $\sigma_n = \sigma_{n-1}$ and at $\sigma_n = \sigma_1$, while for gravity there are $n-1$ poles at $\sigma_n = \sigma_c$ for $c = 1, \ldots, n-1$.

In both cases, at the pole $\sigma_n = \sigma_c$, only the single term with $a = c$ from the sum in the numerator (for gravity $a = c$ in both sums) and the one with $a = c$ from that in the denominator will contribute. The product of the first two factors becomes independent of i hence can be pulled out of the sum, and by summing over the solutions one recovers amplitudes with $n-1$ particles [2]

$$A_n \to \left(\frac{\epsilon_n \cdot k_{n-1}}{k_n \cdot k_{n-1}} + \frac{\epsilon_n \cdot k_1}{k_n \cdot k_1}\right) A_{n-1}, \quad M_n \to \sum_{a=1}^{n-1} \frac{2\epsilon_n \cdot k_a \tilde{\epsilon}_n \cdot k_a}{k_n \cdot k_a} M_{n-1}, \tag{1.7}$$

which is the correct soft behavior [3, 4].

2 Factorization

For the purpose of showing that (0.1) has the correct behavior in any physical factorization channel, it is sufficient to have a look at only the kinematics singularity defined by

$$(k_1 + k_2 + \cdots + k_{n_L})^2 \longrightarrow 0, \tag{2.1}$$

with $2 \leq n_L \leq n-2$, and we denote $L = \{1, \ldots, n_L\}$ and R as its complenent set, with $n_R = n - n_L$. We will only focus on the factorization of Yang-Mills partial amplitudes $A_n(1, \ldots, n)$. That of the gravity amplitudes follows straightforwardly from the results here.

From the study of scattering equations in [2], we have known that upon such singularity, the scattering equations will produce $(n_L - 2)! \times (n_R - 2)!$ singular solutions, in which the punctures of the particles either in L or in R will collapse together. In order to distinguish these singular solutions from the rest and show explicitly the leading behavior of the formula

[2] Here the additional 2 in the soft factor for gravity is due to our convention on the normalization of the polarization vectors such that when restricting to 4-dimensions and using spin-helicity formalism we have $\epsilon^- = \frac{\lambda\tilde{\mu}}{[\tilde{\lambda}\tilde{\mu}]}$ and $\frac{\mu\tilde{\lambda}}{\langle\mu\lambda\rangle}$.

upon them, it is convenient to start by re-defining the variables in the following way

$$\sigma_a = \begin{cases} \frac{s}{u_a}, & a \in L \\ \frac{v_a}{s}, & a \in R \end{cases}, \tag{2.2}$$

where we regard $v_{n-1} = v^*_{n-1}$ as being fixed to a specific value v^*_{n-1}, and leave s as well as the remaining u's and v's as varaibles to be integrated over. With this re-definition, the measure becomes

$$\prod_{a=1}^{n} d\sigma_a = (-1)^{n_L+1} s^{n_L-n_R-1} \frac{v^*_{n-1}}{(\prod u)^2} ds \prod_{a\in L} du_a \prod_{a\in R\backslash\{n-1\}} dv_a. \tag{2.3}$$

where $\prod u$ denotes the product of all u_a with $a \in L$.

Now the SL$(2,\mathbb{C})$ redundancy still exsits, but acts non-trivially on the variables $\{s, u, v\}$ (in particular v^*_{n-1} remains invariant)

$$\begin{aligned} s &\longmapsto s' = v^*_{n-1} \frac{Cv^*_{n-1} + Ds}{Av^*_{n-1} + Bs}, \\ u_a &\longmapsto u'_a = v^*_{n-1} \frac{Cv^*_{n-1} + Ds}{Av^*_{n-1} + Bs} \frac{Cs + Du_a}{As + Bu_a}, \\ v_a &\longmapsto v'_a = v^*_{n-1} \frac{Cv^*_{n-1} + Ds}{Av^*_{n-1} + Bs} \frac{Av_a + Bs}{Cv_a + Ds}, \end{aligned} \tag{2.4}$$

with $AD - BC = 1$. This redundancy can be fixed by further setting three variables to specific values via Faddeev-Popov method (For the convenience of later discussions we choose to fix $\{u_1, u_2, v_n\}$). To be explicit, if we use the abbreviation

$$\begin{aligned} A_n(1,\ldots,n) &= \frac{1}{\text{vol SL}(2,\mathbb{C})} \int \prod_{a=1}^{n} d\sigma_a \mathcal{I}(\{\sigma\}) \\ &= \frac{(-1)^{n_L+1}}{\text{vol SL}(2,\mathbb{C})} \int \frac{s^{n_L-n_R-1} v^*_{n-1}}{(\prod u)^2} ds \prod_{a\in L} du_a \prod_{a\in R\backslash\{n-1\}} dv_a \mathcal{I}(\{\frac{s}{u}, \frac{v}{s}\}), \end{aligned} \tag{2.5}$$

and change the notation of the integrand variables from $\{s, u, v\}$ to $\{s', u', v'\}$, we can insert a "1" into the integrand $\mathcal{I}(\{\frac{s}{u}, \frac{v}{s}\})$, which is

$$\int dAdBdC \prod_{a=1}^{2} \delta(u'_a - v^*_{n-1} \frac{Cv^*_{n-1} + Ds^*}{Av^*_{n-1} + Bs^*} \frac{Cs^* + Du^*_a}{As^* + Bu^*_a}) \delta(v'_n - v^*_{n-1} \frac{Cv^*_{n-1} + Ds^*}{Av^*_{n-1} + Bs^*} \frac{Av^*_n + Bs^*}{Cv^*_n + Ds^*}) J, \tag{2.6}$$

where s^* should be understood as a function of s' via the inverse of (2.4)

$$s^* = s^*(s') = v^*_{n-1} \frac{-As' + Cv^*_{n-1}}{Bs' - Dv^*_{n-1}}, \tag{2.7}$$

so that the Jacobian J in (2.6) is (here we switch s' back to $s^*(s')$)

$$\frac{4s^* u^*_{1,2} v^*_{n-1,n} {v^*_{n-1}}^2 (s^* + s^*BC + ACv^*_{n-1})^3 (-s^{*4} + u^*_1 u^*_2 v^*_{n-1} v^*_n)}{A^2 (Av^*_{n-1} + Bs^*)^3 \prod_{a=1}^{2} (As^* + Bu^*_a)^2 (s^* + s^*BC + ACv^*_n)^2}. \tag{2.8}$$

The purpose of regarding s^* as the inverse transformation from s' is to match this s' with the integration variable in the formula so that $J = J(s')$ can enter as part of the integrand. In order to isolate the gauge directions, we can regard $\{s', u', v'\}$ as transformed from a set of new variables which we again denote as $\{s, u, v\}$, exactly via the relations in (2.4). It is easy to see that now we have the indentification $s = s^*$, and the three delta functions in (2.6) are equivalent to $\delta(u_1 - u_1^*)\delta(u_2 - u_2^*)\delta(v_n - v_n^*)$. Since the Jacobian of those delta functions with respect to $\{u_1^*, u_2^*, v_n^*\}$ is

$$-\frac{{s^*}^3 {v_{n-1}^*}^3 (s^* + s^*BC + ACv_{n-1}^*)^3}{A(Av_{n-1}^* + Bs^*)^3 \prod_{a=1}^2 (As^* + Bu_a^*)^2 (s^* + s^*BC + ACv_n^*)^2}, \tag{2.9}$$

the additional factor (2.6) becomes (with $s = s^*$)

$$-\int \frac{dAdBdC}{A} \delta(u_1 - u_1^*)\delta(u_2 - u_2^*)\delta(v_n - v_n^*) \frac{4u_{1,2}^* v_{n-1,n}^* (-s^4 + u_1^* u_2^* v_{n-1}^* v_n^*)}{s^2 v_{n-1}^*}. \tag{2.10}$$

If given the condition that s is infinitesimal, (2.10) can be approximated to

$$-\int \frac{dAdBdC}{A} \delta(u_1 - u_1^*)\delta(u_2 - u_2^*)\delta(v_n - v_n^*) 4s^{-2} u_{1,2}^* v_{n-1,n}^* u_1^* u_2^* v_n^*. \tag{2.11}$$

On the other hand, since SL$(2, \mathbb{C})$ invariance is establish in the original integrand together with the measure, i.e.

$$\frac{{s'}^{n_L - n_R - 1} v_{n-1}^*}{(\prod u')^2} ds' \prod du_a' \prod dv_a' \mathcal{I}(\{\frac{s'}{u'}, \frac{v'}{s'}\}) = \frac{s^{n_L - n_R - 1} v_{n-1}^*}{(\prod u)^2} ds \prod du_a \prod dv_a \mathcal{I}(\{\frac{s}{u}, \frac{v}{s}\}), \tag{2.12}$$

and we identify the SL$(2, \mathbb{C})$ in the usual way

$$\text{vol SL}(2, \mathbb{C}) = \int \frac{2dAdBdC}{A}, \tag{2.13}$$

then if we integrate out $\{u_1, u_2, v_n\}$ by the delta functions in (2.11), in the domain where s can be constrained to be infinitesimal we obtain the leading behavior of the formula as

$$A_n \longrightarrow (-1)^{n_L} \int ds^2 \prod_{a \in L \backslash \{1,2\}} du_a \prod_{a \in R \backslash \{n-1,n\}} dv_a s^{n_L - n_R - 4} \frac{u_{1,2}^* u_1^* u_2^* v_{n-1,n}^* v_{n-1}^* v_n^*}{(\prod u)^2} \mathcal{I}(\{\frac{s}{u}, \frac{v}{s}\}). \tag{2.14}$$

From now on we will write the fixed values $\{u_1^*, u_2^*, v_{n-1}^*, v_n^*\}$ just as $\{u_1, u_2, v_{n-1}, v_n\}$.

The existence of infinitesimal solutions of s in the factorization limit is guarenteed by the constraints. For convenience, we choose to eliminate the constraints corresponding to particles $\{1, 2, n\}$. We first study constraints corresponding to $a \in R$, which now become

$$\begin{aligned} &\frac{s}{v_a}(s_{a,1} + \cdots + s_{a,n_L}) + s\left(\frac{s_{a,n_L+1}}{v_{a,n_L+1}} + \cdots + \frac{s_{a,n}}{v_{a,n}}\right) + \mathcal{O}(s^3) \\ &\qquad = s\left(\frac{s_{a,I_R}}{v_a} + \frac{s_{a,n_L+1}}{v_{a,n_L+1}} + \cdots + \frac{s_{a,n}}{v_{a,n}}\right) + \mathcal{O}(s^3) = 0, \end{aligned} \tag{2.15}$$

where we define $k_{I_R}^\mu = k_1^\mu + \cdots + k_{n_L}^\mu$. If we dress the constraint a with a factor $\frac{v_a v_{n,a}}{s v_n}$ and sum over the label a from n_L+1 to $n-1$, we may find

$$-k_{I_R}^2 + s^2 \left(F(k,u,v) + \mathcal{O}(s)\right) = 0, \tag{2.16}$$

where F is some rational function of $\{k,u,v\}$ and independent of s. By this constraint, we conclude that upon the kinematics singularity, a subset of solutions are associated with infinitesimal s, which are going to produce the leading terms in the factorization. Since we only care about the leading terms, it is thus justified to do approximations in s and discard the higher order terms everywhere else. So for the δ functions whose labels belong to R, they approximate to

$$\prod_{a\in R\backslash\{n\}} \delta\Big(\sum_{b\neq a} \frac{s_{a,b}}{\sigma_{a,b}}\Big) \longrightarrow s^{-n_R+1} \frac{v_{n-1}v_{n,n-1}}{v_n} \delta(s^2 F - k_{I_R}^2) \prod_{a\in R\backslash\{n-1,n\}} \delta\Big(\sum_{b\in R\cup\{I_R\}\backslash\{a\}} \frac{s_{a,b}}{v_{a,b}}\Big). \tag{2.17}$$

Similarly, one can verify that the remaining δ functions whose labels belong to L become

$$\prod_{a\in L\backslash\{1,2\}} \delta\Big(\sum_{b\neq a} \frac{s_{a,b}}{\sigma_{a,b}}\Big) \longrightarrow \frac{(-1)^{n_L-2} s^{n_L-2}}{\prod_{a\in L\backslash\{1,2\}} u_a^2} \sum_{a\in L\backslash\{1,2\}} \delta\Big(\sum_{b\in L\cup\{I_L\}\backslash\{a\}} \frac{s_{a,b}}{u_{a,b}}\Big). \tag{2.18}$$

Counting the constraints corresponding to the internal particles $\{I_L, I_R\}$ as well but regard them as being deleted from both sides, (2.17) and (2.18) indicate that

$$\begin{aligned}\sigma_{1,2}\sigma_{2,n}\sigma_{n,1}\prod_{a=3}^{n-1}\delta\Big(\sum_{b\neq a}\frac{s_{a,b}}{\sigma_{a,b}}\Big) \longrightarrow &(-1)^{n_L-1}\frac{s^{n_L-n_R-2}}{(\prod u)^2}\delta(s^2F-k_{I_R}^2)\cdot \\ &\cdot u_{1,2}u_1u_2\prod_{a=3}^{I_L}\delta\Big(\frac{s_{a,b}}{u_{a,b}}\Big)v_{n-1,n}v_{n-1}v_n\prod_{a=I_R}^{n-2}\delta\Big(\frac{s_{a,b}}{v_{a,b}}\Big).\end{aligned} \tag{2.19}$$

The Parke-Taylor form may contribute different powers in s depending on the orderings of the labels. It is easy to observe that, in order to produce the most singular term (i.e. the leading term) in the neighborhood of infinitesimal s, we expect that this form splits into two parts, one of which contains all the labels in L while the other all the labels in R. Here we only focus on one such example, $\sigma_{1,2}\cdots\sigma_{n-1,n}\sigma_{n-1}$, which approximates to

$$\frac{1}{\sigma_{1,2}\cdots\sigma_{n,1}} = \frac{(-1)^{n_L}s^{-n_L+n_R+2}(\prod u)^2}{(u_1u_{1,2}\cdots u_{n_L-1,n_L}u_{n_L})(v_{n_L+1}v_{n_L+1,n_L+2}\cdots v_{n-1,n}v_n)} + \mathcal{O}(s^{-n_L+n_R+3}). \tag{2.20}$$

Parke-Taylor forms with labels which are permutations within the set L and R respectively from the above example also produce the same power in s, while those with the other orderings are all subleading and thus do not contribute to the factorization under study.

In order to check the behavior of $\mathrm{Pf}'\Psi$, it is more convenient to re-arrange the rows and columns so that the original $(i+n)^{\text{th}}$ row or column comes right after the original i^{th} row or column (which only results in an overall sign $(-1)^{\frac{n(n-1)}{2}}$). The 2×2 block of $(2a-1)^{\text{th}}$ and $(2a)^{\text{th}}$ rows and $(2b-1)^{\text{th}}$ and $(2b)^{\text{th}}$ columns are

$$\begin{pmatrix} \frac{2k_a\cdot k_b}{\sigma_{a,b}} & \frac{2k_a\cdot \epsilon_b}{\sigma_{a,b}} \\ \frac{2\epsilon_a\cdot k_b}{\sigma_{a,b}} & \frac{2\epsilon_a\cdot \epsilon_b}{\sigma_{a,b}} \end{pmatrix} \tag{2.21}$$

if $a\neq b$, and

$$\begin{pmatrix} 0 & \sum_{c\neq a}\frac{2\epsilon_a\cdot k_c}{\sigma_{a,c}} \\ -\sum_{c\neq a}\frac{2\epsilon_a\cdot k_c}{\sigma_{a,c}} & 0 \end{pmatrix} \tag{2.22}$$

if $a=b$. Now the entire matrix consists of $n\times n$ such 2×2 blocks. And we can study their behavior upon the singularity $s\to 0$. For the blocks (2.21) where $a\neq b$, when both a,b are on the same side, it approximates to

$$-\frac{u_a u_b}{s}\begin{pmatrix} \frac{2k_a\cdot k_b}{u_{a,b}} & \frac{2k_a\cdot \epsilon_b}{u_{a,b}} \\ \frac{2\epsilon_a\cdot k_b}{u_{a,b}} & \frac{2\epsilon_a\cdot \epsilon_b}{u_{a,b}} \end{pmatrix},\quad a,b\in L,\qquad s\begin{pmatrix} \frac{2k_a\cdot k_b}{v_{a,b}} & \frac{2k_a\cdot \epsilon_b}{v_{a,b}} \\ \frac{2\epsilon_a\cdot k_b}{v_{a,b}} & \frac{2\epsilon_a\cdot \epsilon_b}{v_{a,b}} \end{pmatrix},\quad a,b\in R, \tag{2.23}$$

and when they are on different sides, it approximates to

$$-\frac{s}{v_b}\begin{pmatrix} 2k_a\cdot k_b & 2k_a\cdot \epsilon_b \\ 2\epsilon_a\cdot k_b & 2\epsilon_a\cdot \epsilon_b \end{pmatrix},\quad a\in L, b\in R,\qquad \frac{s}{v_a}\begin{pmatrix} 2k_a\cdot k_b & 2k_a\cdot \epsilon_b \\ 2\epsilon_a\cdot k_b & 2\epsilon_a\cdot \epsilon_b \end{pmatrix},\quad a\in R, b\in L. \tag{2.24}$$

For the blocks (2.22) where $a=b$, when $a\in L$ it approximates to (since the block itself is skew-symmetric, we only write out its upper-triangle entries)

$$\begin{pmatrix} 0 & -\frac{u_a}{s}\sum_{c\in L\backslash\{a\}}\frac{2\epsilon_a\cdot k_c u_c}{u_{a,c}}+\mathcal{O}(s) \\ \cdots & 0 \end{pmatrix} \approx -\frac{u_a^2}{s}\begin{pmatrix} 0 & \sum_{c\in L\cup\{I_L\}\backslash\{a\}}\frac{2\epsilon_a\cdot k_c}{u_{a,c}} \\ \cdots & 0 \end{pmatrix}, \tag{2.25}$$

and when $a\in R$ it approximates to

$$\begin{pmatrix} 0 & s(\sum_{c\in L}\frac{2\epsilon_a\cdot k_c}{v_a}+\sum_{c\in R\backslash\{a\}}\frac{2\epsilon_a\cdot k_c}{v_{a,c}})+\mathcal{O}(s^3) \\ \cdots & 0 \end{pmatrix} \approx s\begin{pmatrix} 0 & \sum_{c\in R\cup\{I_R\}\backslash\{a\}}\frac{2\epsilon_a\cdot k_c}{v_{a,c}} \\ \cdots & 0 \end{pmatrix}. \tag{2.26}$$

From (2.23), (2.25) and (2.26), we see that apart from the the overall factors these blocks already go in to the forms that we expect for the Pfaffians in the two subampitudes, except for the fact that we haven't yet seen the appearance of entries corresponding to the internal particle.

From now on we would like to change the notation of entries. Since within each block the entries are associated with only particular labels a and b, for convenience we can denote each block just by (a,b). If we need to further distinguish specific entries, we will write $a^{(k)}$ and $a^{(\epsilon)}$ according to whether it denotes the first row or the second row (resp. column). And further we define the ordering of these labels as

$$1^{(k)}<1^{(\epsilon)}<2^{(k)}<\cdots<(n-1)^{(\epsilon)}<n^{(k)}<n^{(\epsilon)}, \tag{2.27}$$

so that any label in L is regarded as smaller than any label in R.

Since the leading order is $1/s$ only in the first situation of (2.23) and in (2.25) and s in all other situations, we directly see that the leading terms of the Pfaffian $\mathrm{Pf}'(\Psi)$ must avoid the contribution from the entries that mix the left labels and the right labels as much as possible. In computing the Pfaffian, let's make a convention to delete the entries with label $\{1^{(k)}, n^{(k)}\}$, so that the number of remaining labels in L and that of R are both odd

$$\begin{aligned} L^{(k,\epsilon)} &= \{1^{(\epsilon)}, 2^{(k)}, 2^{(\epsilon)}, \dots, n_L^{(k)}, n_L^{(\epsilon)}\}, \\ R^{(k,\epsilon)} &= \{(n_L+1)^{(k)}, (n_L+1)^{(\epsilon)}, \dots, (n-1)^{(k)}, (n-1)^{(\epsilon)}, n^{(\epsilon)}\}. \end{aligned} \tag{2.28}$$

Recall the definition of Pfaffian, this means that in any leading-order term, we always have exactly one factor that comes from (2.24). Denote the label of this entry as (a^*, b^*) with $a^* \in L$ and $b^* \in R$. If we also introduce the labels for the interal particle with the order $\{I_L^{(k)}, I_L^{(\epsilon)}, I_R^{(k)}, I_R^{\epsilon}\}$, all of which are regarded as greater than any labels in $L^{(k,\epsilon)}$ and smaller than any labels in $R^{(k,\epsilon)}$, the sign attached to any leading-order term can then be expressed into the form [3]

$$\mathrm{sign}(L^{(k,\epsilon)}\backslash\{a^*\}, a^*, b^*, R^{(k,\epsilon)}\backslash\{b^*\}) = \mathrm{sign}(L^{(k,\epsilon)}\backslash\{a^*\}, a^*, I_L^{(\epsilon)}) \cdot \mathrm{sign}(I_R^{(\epsilon)}, b^*, R^{(k,\epsilon)}\backslash\{b^*\}). \tag{2.29}$$

Here a^* (b^*) can exhaust all labels in $L^{(k,\epsilon)}$ ($R^{(k,\epsilon)}$), and for a fixed pair (a^*, b^*), the leading terms exhaust all the allowed permutations within $L^{(k,\epsilon)}\backslash\{a^*\}$ and within $R^{(k,\epsilon)}\backslash\{b^*\}$ respectively, so we already see that all the leading terms have the correct permutations together with the correct sign needed in order to obtain a product of two smaller Pfaffians corresponding to the sets $\{1^K\} \cup L^{(k,\epsilon)} \cup \{I_L^{(k)}, I_L^{(\epsilon)}\}$ (by deleting $\{1^{(k)}, I_L^{(k)}\}$) and $\{I_R^{(k)}, I_R^{(\epsilon)}\} \cup R^{(k,\epsilon)} \cup \{n^{(k)}\}$ (by deleting $\{I_R^{(k)}, n^{(k)}\}$) respectively.

In order to see the exact decomposition of the original Pfaffian, notice that most of the factors in any leading term are already in the form as coming from either of the two smaller Pfaffians, except for the remaining factor from (2.24). For such factor, let's denote it universally as $e_{a^*} \cdot e_{b^*}$ (i.e. e can be either k or ϵ). Then we can modify this factor into

$$\frac{2s}{v_{b^*}} e_{a^*} \cdot e_{b^*} = \frac{2s}{v_{b^*}} e^{\mu}_{a^*} e^{\nu}_{b^*} \eta_{\mu\nu} = \frac{4s}{v_{b^*}} e^{\mu}_{a^*} e^{\nu}_{b^*} (\sum_{\epsilon_I} \epsilon_{I_L,\mu} \epsilon_{I_R,\nu} + \frac{k_{I_L,\mu} k_{I_R,\nu}}{k_I^2}) = u_{a^*} \sum_{\epsilon_I} \frac{2e_{a^*} \cdot \epsilon_{I_L}}{u_{a^*}} \frac{2\epsilon_{I_R} \cdot e_{b^*}}{v_{b^*}}. \tag{2.30}$$

In the above, the last equality by itself is not true, but since if we get exactly the product of two sub-amplitudes in the end, the k_I just represents the additional term in some gauge transformation, which is zero, so we remove it right away. Here ϵ_{I_L} and ϵ_{I_R} essentially denotes the same polarization vector of the internal particle (but with opposite helicities, and so $e_{I_L} = (e_{I_R})^*$), and we write them in the way to keep notations coherent for the left part and the right part respetively [4]. One can immediately observe that the expression in the end

[3] According to the original definition a^*, b^* are inserted into some odd position in $L^{(k,\epsilon)}\backslash\{a^*\}$, but since they come in pair, so we can freely move them to the end of $L^{(k,\epsilon)}\backslash\{a^*\}$.

[4] The additional factor 2 in the second equality is due to our convention on ϵ; see the footnote at the end of the section on soft limits.

of (2.30) is in the desired form as coming from the entries $I_L^{(\epsilon)}$ and $I_R^{(\epsilon)}$. Now according to the structure of (2.29) each leading term has the order $s^{-(n_L-1)+1+(n_R-1)} = s^{-n_L+n_R+1}$. From (2.23), (2.25) and (2.30) we see that each label $a \in L^{(k,\epsilon)}$ is also accompanied by an extra factor u_a. Hence we conclude that to the leading order the original Pfaffian approximates to (ignore the overall sign)

$$\begin{aligned} \mathrm{Pf}'(\Psi) = \frac{1}{\sigma_{1,n}}\mathrm{Pf}\Psi^{1n}_{1n} \longrightarrow & \frac{s^{-n_L+n_R+2}(\prod u)^2}{u_1 v_n}\mathrm{Pf}(\Psi_L)^{1I_L}_{1I_L}\mathrm{Pf}(\Psi_R)^{1I_R}_{1I_R} \\ =& s^{-n_L+n_R+2}(\prod u)^2\mathrm{Pf}'(\Psi_L)\mathrm{Pf}'(\Psi_R). \end{aligned} \tag{2.31}$$

Collecting the resutls in (2.14), (2.19), (2.20) and (2.31) together, it is easy to see that if we regard the factors $\{u_1, u_2, v_{n-1}, v_n\}$ therein as $\{u_1 - u_{I_L}, u_2 - u_{I_L}, v_{n-1} - v_{n_R}, v_n - v_{n_R}\}$, with u_{I_L} and v_{I_R} as punctures for the internal particles on the left part and right part, which are fixed to be zero, then we have an emergent $\mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C})$ redundancy in these leading terms, acting on $L\cup\{u_{I_L}\}$ and $R\cup\{v_{I_R}\}$ respectively. When completing the integration over s^2, all the extra factors cancel out and only those that enter into the formula for the two smaller amplitudes remain. Thus we conclude that in the factorization limit (2.1) the leading terms in the original amplitude factorize as

$$A_n(1,\ldots,n) \longrightarrow \sum_{\epsilon_I} A_{n_L+1}(1,\ldots,n_L,I_L)\frac{1}{k_I^2}A_{n_R+1}(I_R,n_L+1,\ldots,n). \tag{2.32}$$

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