Supplementary Document

Scattering of Massless Particles in Arbitrary Dimension: Soft Limits and Factorizations

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Abstract: This note supplements the recent paper [1] of the current authors by providing a detailed proof for both the soft limits and the factorizations of the formula proposed therein.

Contents

We study in turn the soft limit and factorizations on physical poles of the formulas for Yang-Mills and gravity tree-level S-matrices in any dimensions,

$$\begin{aligned} A_n =& \frac{1}{\mathrm{vol}\,\mathrm{SL}(2,\mathbb{C})} \int \frac{d^{\,n}\sigma}{\sigma_{12}\cdots\sigma_{n1}} \prod_a{}' \delta(\sum_{b\neq a} \frac{k_a\cdot k_b}{\sigma_{ab}})\,\mathrm{Pf}'\Psi(k,\epsilon,\sigma). \\ M_n =& \frac{1}{\mathrm{vol}\,\mathrm{SL}(2,\mathbb{C})} \int d^{\,n}\sigma \prod_a{}' \delta(\sum_{b\neq a} \frac{k_a\cdot k_b}{\sigma_{ab}})\,\mathrm{Pf}'\Psi(k,\epsilon,\sigma)\,\mathrm{Pf}'\Psi(k,\tilde{\epsilon},\sigma). \end{aligned} \tag{0.1}$$

To simplify notations, we also denote $\tilde{\Psi} = \Psi(k,\tilde{\epsilon},\sigma)$. Both formulas can be written in a form where all the integrals have been performed

$$A_n = \sum_{\{\sigma\}\in \text{ solutions}} \frac{1}{\sigma_{12}\cdots\sigma_{n1}} \frac{\mathrm{Pf}'\Psi(k,\epsilon,\sigma)}{\det'\Phi} \tag{0.2}$$

and

$$M_n = \sum_{\{\sigma\}\in \text{ solutions}} \frac{\mathrm{Pf}'\Psi(k,\epsilon,\sigma)\,\mathrm{Pf}'\Psi(k,\tilde{\epsilon},\sigma)}{\det'\Phi}. \tag{0.3}$$

The definition of the matrix Ψ and the reduced Pfaffian as well as its consistency (i.e. gauge invariance and permutation invariance) have been discussed in detail in [1].

1 Soft Limit

As discussed in detail in [2], when we take the soft limit $k_n \to 0$, $k_n \equiv \varepsilon \hat{k}_n \to 0$, to leading order in ε, $n-1$ of the scattering equations become identical to those of a system with $n-1$ particles. The last equation

$$\sum_{b\neq n} \frac{k_n\cdot k_b}{\sigma_n - \sigma_b} = 0 \tag{1.1}$$

becomes a polynomial for σ_n of degree $n-3$ (due to momentum conservation). With the choice that σ_n unfixed and the n^{th} equation included, the measure and delta functions split into the $(n-1)$-particle part and the part for the n^{th} particle,

$$\frac{d^{\,n}\sigma}{\mathrm{vol}\,\mathrm{SL}(2,\mathbb{C})} \prod_{a=1}^{n}{}' \delta(\sum_{b\neq a} \frac{k_a\cdot k_b}{\sigma_{a,b}}) \to \frac{d^{\,n-1}\sigma}{\mathrm{vol}\,\mathrm{SL}(2,\mathbb{C})} \prod_{a=1}^{n-1}{}' \delta(\sum_{b\neq a} \frac{k_a\cdot k_b}{\sigma_{a,b}}) d\sigma_n \delta(\sum_{a=1}^{n-1} \frac{k_n\cdot k_a}{\sigma_{n,a}})\,. \tag{1.2}$$

The kinematic-dependence of the integrand of Yang-Mills/gravity amplitude is encoded in the reduced Pfaffian $\mathrm{Pf}'\Psi$, and only the n^{th} column and row of Ψ depend on k_n under consideration. The Pfaffian of a $2m \times 2m$ matrix E satisfies a recursion relation of the form (ignoring the overall sign)

$$\mathrm{Pf}(E) = \sum_{q=1}^{2m} (-1)^q e_{pq} \mathrm{Pf}(E_{pq}^{pq}). \tag{1.3}$$

Using this formula to expand $\mathrm{Pf}\Psi_{ij}^{ij}$ (assuming $i \neq n, j \neq n$) setting $p = n$, and note that Ψ_{na} for $a = 1, \ldots, 2n{-}1$ are proportional to k_n (except $\Psi_{nn} = 0$) while $\Psi_{n,2n} = \sum_{a=1}^{n-1} \frac{\epsilon_n \cdot k_a}{\sigma_{n,a}}$, thus one finds that in the soft limit only one term contributes and gives

$$\mathrm{Pf}\Psi_{ij}^{ij} \to C_{nn} \mathrm{Pf}\Psi_{ijn(2n)}^{ijn(2n)} = \sum_{a=1}^{n-1} \frac{\epsilon_n \cdot k_a \prod_{b \neq a} \sigma_{n,b}}{\prod_{c=1}^{n} \sigma_{n,c}} \mathrm{Pf}\Psi_{n-1}{}_{ij}^{ij}, \tag{1.4}$$

where in the second equality one uses the nice fact that $\mathrm{Pf}\Psi_{ijn(2n)}^{ijn(2n)}$ is independent of k_n and ϵ_n and gives rise to $\mathrm{Pf}'\Psi_{n-1}$, i.e., the reduced Pfaffian for $n-1$ particles. For Yang-Mills amplitudes, using the explicit formula (0.2) in the soft limit one finds

$$A_n \to \sum_{i=1}^{(n-4)!} \oint_\Gamma d\sigma_n \frac{\sum_{a \neq n} \epsilon_n \cdot k_a \prod_{b \neq a} \sigma_{n,b}^{(i)}}{\sum_{a \neq n} k_n \cdot k_a \prod_{b \neq a} \sigma_{n,b}^{(i)}} \frac{\sigma_{n-1,1}^{(i)}}{\sigma_{n-1,n}^{(i)} \sigma_{n,1}^{(i)}} \frac{1}{\sigma_{1,2}^{(i)} \cdots \sigma_{n-1,1}^{(i)}} \frac{\mathrm{Pf}'\Psi_{n-1}^{(i)}}{\det'\Phi_{n-1}^{(i)}}, \tag{1.5}$$

where the sum is over $(n{-}4)!$ solutions of the equations for $n{-}1$ particles and all $\sigma_a^{(i)}$'s with $a \in \{1, 2, \ldots, n-1\}$ are taken to be evaluated on the i^{th} solution. Since σ_a's are taken to be complex numbers in our formulas, the delta functions imposing the scattering equations are in fact poles and all our integrals are contour integrals; here the contour Γ for the integral over σ_n is defined to encircle the $n-3$ zeroes of the first factor in the denominator. Similarly for gravity amplitudes using (0.3) one finds

$$M_n \to \sum_{i=1}^{(n-4)!} \oint_\Gamma d\sigma_n \frac{\sum_{a \neq n} \epsilon_n \cdot k_a \prod_{b \neq a} \sigma_{n,b}^{(i)}}{\sum_{a \neq n} k_n \cdot k_a \prod_{b \neq a} \sigma_{n,b}^{(i)}} \frac{\sum_{a \neq n} \epsilon_n \cdot k_a \prod_{b \neq a} \sigma_{n,b}^{(i)}}{\prod_{c=1}^{n-1} \sigma_{n,c}^{(i)}} \frac{\mathrm{Pf}'\Psi_{n-1}^{(i)} \mathrm{Pf}'\tilde{\Psi}_{n-1}^{(i)}}{\det'\Phi_{n-1}^{(i)}}. \tag{1.6}$$

Let us study the pole at infinity: as $\sigma_n \to \infty$, the first factor goes like a constant since the polynomial in the numerator is also of degree $n{-}3$, and in both cases the second factor goes like $1/\sigma_n^2$. Hence for both Yang-Mills and gravity there is no contribution at infinity, and using residue theorems the contour can be deformed to enclose the poles of the second factor in both cases. For Yang-Mills, there are only two poles at $\sigma_n = \sigma_{n-1}$ and at $\sigma_n = \sigma_1$, while for gravity there are $n{-}1$ poles at $\sigma_n = \sigma_c$ for $c = 1, \ldots, n{-}1$.

In both cases, the residue at $\sigma_n = \sigma_c$ is trivial to compute, as only a single term with $a = c$ from the sum in the numerator (for gravity $a = c$ in both sums) and the one with $a = c$ from that in the denominator contribute. The product of the first two factors becomes independent of i hence can be pulled out of the sum, and by summing over the solutions one obtains amplitudes with $n-1$ particles

$$A_n \to \left(\frac{\epsilon_n \cdot k_{n-1}}{k_n \cdot k_{n-1}} + \frac{\epsilon_n \cdot k_1}{k_n \cdot k_1}\right) A_{n-1}, \quad M_n \to \sum_{a=1}^{n-1} \frac{\epsilon_n \cdot k_a \tilde{\epsilon}_n \cdot k_a}{k_n \cdot k_a} M_{n-1}, \tag{1.7}$$

which is the correct soft behavior [3, 4].

2 Factorization

For the purpose of showing that (0.1) has the correct behavior in any physical factorization channel, it is sufficient to have a look at only the kinematics singularity defined by

$$(k_1 + k_2 + \cdots + k_{n_L})^2 \longrightarrow 0, \tag{2.1}$$

with $2 \leq n_L \leq n-2$, and we denote $L = \{1, \ldots, n_L\}$ and R as its complenent set, with $n_R = n - n_L$. We will only focus on the factorization of Yang-Mills amplitudes A_n. That of the gravity amplitudes follows straightforwardly from the results here.

From the study of scattering equations in [2], we have known that upon such singularity, the scattering equations will produce $(n_L-2)! \times (n_R-2)!$ singular solutions. When we choose to fix $\{\sigma_1, \sigma_2, \sigma_n\}$ for the SL$(2,\mathbb{C})$ redundancy, it is convenient to do the following re-definitions of the σ's in order to see the singularity explicitly

$$\sigma_a = \begin{cases} \frac{s}{u_a}, & a \in L \\ \frac{v_a}{s}, & a \in R \end{cases}, \tag{2.2}$$

where we take s as a variable to be integrated over, while $\{u_1, u_2, v_{n-1}, v_n\}$ are fixed. It is a simple exercise to show that the integration measure becomes (in the following analysis we always ignore the overall sign)

$$\frac{\prod d\sigma}{\text{vol SL}(2,\mathbb{C})} = s^{n_L - n_R - 4} \frac{u_{1,2} u_1 u_2 v_{n-1,n} v_{n-1} v_n}{(\prod u)^2} ds^2 \prod_{a \in L \backslash \{1,2\}} du_a \prod_{a \in R \backslash \{n-1,n\}} dv_a, \tag{2.3}$$

where $\prod u$ denotes the product of all u_a with $a \in L$. If s is infinitesimal, one can also show that the Parke-Taylor[1] form approximates to

$$\frac{1}{\sigma_{1,2} \cdots \sigma_{n,1}} = \frac{s^{-n_L+n_R+2} (\prod u)^2}{(u_1 u_{1,2} \cdots u_{n_L-1,n_L} u_{n_L})(v_{n_L+1} v_{n_L+1,n_L+2} \cdots v_{n-1,n} v_n)} + \mathcal{O}(s^{-n_L+n_R+3}). \tag{2.4}$$

From (2.3) and (2.4), we see that if we consider that there are two additional variables but fixed to be zero $u_{I_L} = v_{I_R} = 0$, then at the leading order in s the measure together with the

[1]Here we only study the planar amplitudes that do factorize. For the other planar amplitudes, a simple counting of the order in s of the Parke-Taylor form directly indicates that they should be sub-leading in the factorization.

Parke-Taylor form is invariant under $\mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C})$ which act on the set $L\cup\{I_L\}$ and the set $R\cup\{I_R\}$ respectively. Hence we have

$$\frac{\prod d\sigma}{\text{vol SL}(2,\mathbb{C})}\longrightarrow\frac{ds^2}{s^2}\frac{du_{I_L}\prod du}{\text{vol SL}(2,\mathbb{C})\cdot u_{1,2}\cdots u_{I_L,1}}\frac{dv_{I_R}\prod dv}{\text{vol SL}(2,\mathbb{C})\cdot v_{n_L+1,n_L+2}\cdots v_{I_R,n_L+1}}. \tag{2.5}$$

In the constraints, we choose to eliminate those corresponding to particles $\{1,2,n\}$. We first study constraints corresponding to $a\in R$, which now become

$$\begin{aligned}\frac{s}{v_a}\left(k_a\cdot k_1+\cdots+k_a\cdot k_{n_L}\right)+s\left(\frac{k_a\cdot k_{n_L+1}}{v_{a,n_L+1}}+\cdots+\frac{k_a\cdot k_n}{v_{a,n}}\right)+\mathcal{O}(s^3)\\ =s\left(\frac{k_a\cdot k_{I_R}}{v_a}+\frac{k_a\cdot k_{n_L+1}}{v_{a,n_L+1}}+\cdots+\frac{k_a\cdot k_n}{v_{a,n}}\right)+\mathcal{O}(s^3)=0,\end{aligned} \tag{2.6}$$

where we define $p^\mu_{I_R}=p^\mu_1+\cdots+p^\mu_{n_L}$. If we dress the constraint a with a factor $\frac{v_a v_{n,a}}{s v_n}$ and sum over the label a from n_L+1 to $n-1$, we may find

$$-\frac{1}{2}p^2_{I_R}+s^2\left(F(k,u,v)+\mathcal{O}(s)\right)=0, \tag{2.7}$$

where F is some rational function of $\{k,u,v\}$ and independent of s. By this constraint, we conclude that upon the kinematics singularity, a subset of solutions are associated with infinitesimal s, which are going to produce the leading terms in the factorization. This also confirms the validity of (2.5). Since we only care about the leading terms, it is thus justified to do approximations in s and discard the higher order terms everywhere else. So for the δ functions whose labels belong to R, they approximate to

$$\prod_{a\in R\setminus\{n\}}\delta(\sum_{b\neq a}\frac{k_a\cdot k_b}{\sigma_{a,b}})\longrightarrow s^{-n_R+1}\frac{v_{n-1}v_{n,n-1}}{v_n}\delta(-\frac{1}{2}p^2_{I_R}+s^2F)\prod_{a\in R\setminus\{n-1,n\}}\delta(\sum_{b\in R\cup\{I_R\}\setminus\{a\}}\frac{k_a\cdot k_b}{v_{a,b}}). \tag{2.8}$$

Similarly, one can verify that the remaining δ functions whose labels belong to L become

$$\prod_{a\in L\setminus\{1,2\}}\delta(\sum_{b\neq a}\frac{k_a\cdot k_b}{\sigma_{a,b}})\longrightarrow\frac{s^{n_L-2}}{\prod_{a\in L\setminus\{1,2\}}u_a^2}\sum_{a\in L\setminus\{1,2\}}\delta(\sum_{b\in L\cup\{I_L\}\setminus\{a\}}\frac{k_a\cdot k_b}{u_{a,b}}). \tag{2.9}$$

Counting the constraints corresponding to the internal particles $\{I_L,I_R\}$ as well but regard them as being chosen to deleted from both sides, (2.8) and (2.9) indicate that

$$\prod{}'\delta(\sum_{b\neq a}\frac{k_a\cdot k_b}{\sigma_{a,b}})\longrightarrow\frac{s^{n_L-n_R-2}}{(\prod u)^2}\delta(-\frac{1}{2}p^2_{I_R}+s^2F)\prod_{L\cup\{I_L\}}{}'\delta(\frac{k_a\cdot k_b}{u_{a,b}})\prod_{R\cup\{I_R\}}{}'\delta(\frac{k_a\cdot k_b}{v_{a,b}}). \tag{2.10}$$

In order to check the behavior of $\mathrm{Pf}'\Psi$, it is more convenient to re-arrange the rows and columns so that the original $(i+n)^{\text{th}}$ row or column comes right after the original i^{th} row or column (which only results in an irrelavent overall sign). The 2×2 block of $(2a-1)^{\text{th}}$ and $(2a)^{\text{th}}$ rows and $(2b-1)^{\text{th}}$ and $(2b)^{\text{th}}$ columns are

$$\begin{pmatrix}\frac{k_a\cdot k_b}{\sigma_{a,b}} & \frac{k_a\cdot\epsilon_b}{\sigma_{a,b}}\\ \frac{\epsilon_a\cdot k_b}{\sigma_{a,b}} & \frac{\epsilon_a\cdot\epsilon_b}{\sigma_{a,b}}\end{pmatrix} \tag{2.11}$$

if $a \neq b$, and

$$\begin{pmatrix} 0 & \sum_{c\neq a} \frac{\epsilon_a\cdot k_c}{\sigma_{a,c}} \\ -\sum_{c\neq a} \frac{\epsilon_a\cdot k_c}{\sigma_{a,c}} & 0 \end{pmatrix} \tag{2.12}$$

if $a = b$. Now the entire matrix is consisted of $n \times n$ such 2×2 blocks. And we can study their behavior upon the singularity $s \to 0$. For the blocks (2.11) where $a \neq b$, when both a, b are on the same side, it approximates to

$$-\frac{u_a u_b}{s}\begin{pmatrix} \frac{k_a\cdot k_b}{u_{a,b}} & \frac{k_a\cdot \epsilon_b}{u_{a,b}} \\ \frac{\epsilon_a\cdot k_b}{u_{a,b}} & \frac{\epsilon_a\cdot \epsilon_b}{u_{a,b}} \end{pmatrix}, \quad a,b\in L, \qquad s\begin{pmatrix} \frac{k_a\cdot k_b}{v_{a,b}} & \frac{k_a\cdot \epsilon_b}{v_{a,b}} \\ \frac{\epsilon_a\cdot k_b}{v_{a,b}} & \frac{\epsilon_a\cdot \epsilon_b}{v_{a,b}} \end{pmatrix}, \quad a,b\in R, \tag{2.13}$$

and when they are on different sides, it approximates to

$$-\frac{s}{v_b}\begin{pmatrix} k_a\cdot k_b & k_a\cdot \epsilon_b \\ \epsilon_a\cdot k_b & \epsilon_a\cdot \epsilon_b \end{pmatrix}, \quad a\in L, b\in R, \qquad \frac{s}{v_a}\begin{pmatrix} k_a\cdot k_b & k_a\cdot \epsilon_b \\ \epsilon_a\cdot k_b & \epsilon_a\cdot \epsilon_b \end{pmatrix}, \quad a\in R, b\in L. \tag{2.14}$$

For the blocks (2.12) where $a = b$, when $a \in L$ it approximates to (since the block itself is skew-symmetric, we only write out its upper-triangle entries)

$$\begin{pmatrix} 0 & -\frac{u_a}{s}\sum_{c\in L\setminus\{a\}} \frac{\epsilon_a\cdot k_c u_c}{u_{a,c}} + \mathcal{O}(s) \\ \cdots & 0 \end{pmatrix} \approx -\frac{u_a^2}{s}\begin{pmatrix} 0 & \sum_{c\in L\cup\{I_L\}\setminus\{a\}} \frac{\epsilon_a\cdot k_c}{u_{a,c}} \\ \cdots & 0 \end{pmatrix}, \tag{2.15}$$

and when $a \in R$ it approximates to

$$\begin{pmatrix} 0 & s(\sum_{c\in L} \frac{\epsilon_a\cdot k_c}{v_a} + \sum_{c\in R\setminus\{a\}} \frac{\epsilon_a\cdot k_c}{v_{a,c}}) + \mathcal{O}(s^3) \\ \cdots & 0 \end{pmatrix} \approx s\begin{pmatrix} 0 & \sum_{c\in R\cup\{I_R\}\setminus\{a\}} \frac{\epsilon_a\cdot k_c}{v_{a,c}} \\ \cdots & 0 \end{pmatrix}. \tag{2.16}$$

From (2.13), (2.15) and (2.16), we see that apart from the the overall factors these blocks already go in to the forms that we expect for the Pfaffians in the two subampitudes, except for the fact that we haven't yet seen the appearance of entries corresponding to the internal particle.

From now on we would like to change the notation of entries. Since within each block the entries are associated with only particular labels a and b, for convenience we can denote each block just by (a, b). If we need to further distinguish specific entries, we will write $a^{(k)}$ and $a^{(\epsilon)}$ according to whether it denotes the first row or the second row (resp. column). And further we define the ordering of these labels as

$$1^{(k)} < 1^{(\epsilon)} < 2^{(k)} < \cdots < (n-1)^{(\epsilon)} < n^{(k)} < n^{(\epsilon)}, \tag{2.17}$$

so that any label in L is regarded as smaller than any label in R.

Since the leading order is $1/s$ only in the first situation of (2.13) and in (2.15) and s in all other situations, we directly see that the leading terms of the Pfaffian $\mathrm{Pf}'(\Psi)$ must avoid the contribution from the entries that mix the left labels and the right labels as much as possible. In computing the Pfaffian, let's make a convention to delete the entries with label $\{1^{(k)}, n^{(k)}\}$, so that the number of remaining labels in L and that of R are both odd

$$\begin{aligned} L^{(k,\epsilon)} &= \{1^{(\epsilon)}, 2^{(k)}, 2^{(\epsilon)}, \ldots, n_L^{(k)}, n_L^{(\epsilon)}\}, \\ R^{(k,\epsilon)} &= \{(n_L+1)^{(k)}, (n_L+1)^{(\epsilon)}, \ldots, (n-1)^{(k)}, (n-1)^{(\epsilon)}, n^{(\epsilon)}\}. \end{aligned} \tag{2.18}$$

Recall the definition of Pfaffian, this means that in any leading-order term, we always have exactly one factor that comes from (2.14). Denote the label of this entry as (a^*, b^*) with $a^* \in L$ and $b^* \in R$. If we also introduce the labels for the interal particle with the order $\{I_L^{(k)}, I_L^{(\epsilon)}, I_R^{(k)}, I_R^{\epsilon}\}$, all of which are regarded as greater than any labels in $L^{(k,\epsilon)}$ and smaller than any labels in $R^{(k,\epsilon)}$, the sign attached to any leading-order term can then be expressed into the form [2]

$$\text{sign}(L^{(k,\epsilon)}\backslash\{a^*\}, a^*, b^*, R^{(k,\epsilon)}\backslash\{b^*\}) = \text{sign}(L^{(k,\epsilon)}\backslash\{a^*\}, a^*, I_L^{(\epsilon)}) \cdot \text{sign}(I_R^{(\epsilon)}, b^*, R^{(k,\epsilon)}\backslash\{b^*\}). \tag{2.19}$$

Here a^* (b^*) can exhaust all labels in $L^{(k,\epsilon)}$ $(R^{(k,\epsilon)})$, and for a fixed pair (a^*, b^*), the leading terms exhaust all the allowed permutations within $L^{(k,\epsilon)}\backslash\{a^*\}$ and within $R^{(k,\epsilon)}\backslash\{b^*\}$ respectively, so we already see that all the leading terms have the correct permutations together with the correct sign needed in order to obtain a product of two smaller Pfaffians corresponding to the sets $\{1^K\} \cup L^{(k,\epsilon)} \cup \{I_L^{(k)}, I_L^{(\epsilon)}\}$ (by deleting $\{1^{(k)}, I_L^{(k)}\}$) and $\{I_R^{(k)}, I_R^{(\epsilon)}\} \cup R^{(k,\epsilon)} \cup \{n^{(k)}\}$ (by deleting $\{I_R^{(k)}, n^{(k)}\}$) respectively.

In order to see the exact decomposition of the original Pfaffian, notice that most of the factors in any leading term are already in the form as coming from either of the two smaller Pfaffians, except for the remaining factor from (2.14). For such factor, let's denote it universally as $e_{a^*} \cdot e_{b^*}$ (i.e. e can be either k or ϵ). Then ignoring the overall sign, we can modify this factor into

$$\frac{s}{v_{b^*}} e_{a^*} \cdot e_{b^*} = \frac{s}{v_{b^*}} e_{a^*}^{\mu} e_{b^*}^{\nu} \eta_{\mu\nu} = \frac{s}{v_{b^*}} e_{a^*}^{\mu} e_{b^*}^{\nu} \Big(\sum_{\epsilon_I} \epsilon_{I_L,\mu} \epsilon_{I_R,\nu} + \frac{k_{I_L,\mu} k_{I_R,\nu}}{k_I^2}\Big) = s u_{a^*} \sum_{\epsilon_I} \frac{e_{a^*} \cdot \epsilon_{I_L}}{u_{a^*}} \frac{\epsilon_{I_R} \cdot e_{b^*}}{v_{b^*}}. \tag{2.20}$$

In the above, the last identity by itself is not true, but since if we get exactly the product of two sub-amplitudes in the end, the k_I just represents the additional term in some gauge transformation, which is zero, so we remove it right away. Here ϵ_{I_L} and ϵ_{I_R} essentially denotes the same polarization vector of the internal particle (but with opposite helicities), and we write them in the way to keep notations coherent for the left part and the right part respetively. One can immediately observe that the expression in the end of (2.20) is in the desired form as coming from the entries $I_L^{(\epsilon)}$ and $I_R^{(\epsilon)}$. Now since according to the structure of (2.19) each leading term has the order $s^{-(n_L-1)+1+(n_R-1)} = s^{-n_L+n_R+1}$, and each label $a \in L^{(k,\epsilon)}$ is accompanied by an extrac factor u_a, we hence conclude that to the leading order

[2] According to the original definition a^*, b^* are inserted into some odd position in $L^{(k,\epsilon)}\backslash\{a^*\}$, but since they come in pair, so we can freely move them to the end of $L^{(k,\epsilon)}\backslash\{a^*\}$.

the original Pfaffian approximates to (ignore the overall sign)

$$\begin{aligned}\mathrm{Pf}'(\Psi) = \frac{2}{(\sigma_{1,n})}\mathrm{Pf}(\Psi^{1n}_{1n}) \longrightarrow & \frac{2s^{-n_L+n_R+2}(\prod u)^2}{u_1 v_n}\mathrm{Pf}((\Psi_L)^{1I_L}_{1I_L})\mathrm{Pf}((\Psi_R)^{1I_R}_{1I_R}) \\ = & \frac{1}{2}s^{-n_L+n_R+2}(\prod u)^2 \mathrm{Pf}'(\Psi_L)\mathrm{Pf}'(\Psi_R).\end{aligned} \tag{2.21}$$

Collecting the resutls in (2.5), (2.10) and (2.21) together, it is easy to check the validity of the $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$ redundancies acting on the left and the right part respectively. When completing the integration over s^2, we conclude that in the factorization limit (2.1) the leading terms in the original amplitude factorize as

$$A_n(1,\ldots,n) \longrightarrow \sum_{\epsilon_I} A_{n_L+1}(1,\ldots,n_L,I_L)\frac{1}{P_I^2}A_{n_R+1}(I_R,n_L+1,\ldots,n). \tag{2.22}$$

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